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Construction of Conic Sections whose Elements Are in Part Imaginary

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Construction
of Conic Sections whose Elements are in part Imaginary

by

Neta Hannum

Thesis for the Degree of Bachelor of Arts in Mathematics in the College of Science of the University of Illinois

1905

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HEAD OF DEPARTMENT OF Mathematics

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Chapter I.

Introduction.

In attempting to construct a conic synthetically by means of the straight edge and compass, we are necessarily restricted to finding either points of the conic or lines tangent to the conic; that is, we construct the conic as the locus of a moving point or as the envelope of a moving line. Five such elements, either points or lines, must be given in order to construct the conic, as we know from analytical geometry. Moreover, these elements must be independent of each other.

The problem of constructing a conic when all five elements are real and of the same kind, that is all five are points or all five are lines, is readily solved, as we know, by the aid of Brianchon's and Pascal's Theorems. If, however, the elements are not all of the same

kind, if some of them are imaginary, the problem becomes more difficult, and it is only within comparatively recent times that such solutions have been given.

The first synthetic construction of a conic, when some of the given elements are imaginary was given by Carl Rossfeld and was published in 1882 as a doctor's thesis. It seems that Otto Stolz has also busied himself with this problem, though he has not published his results. Constructions based on his work, and therefore essentially his, have, however, been published by F. Ruth: "Beitag zur Construction der Kegelschnitte aus Imaginaren Elementen" [Monatsheft fur Mathematik, III, 1890, pp. 81-86] and by F. S Path: "Lineale Construction von Kegelschnitte aus theilweise Imaginaren Elementen" [Monatsheft fur Mathematick, I, 1890, pp. 237-246]. Professor Holgate of Northwestern has worked on the problem and gave some of his results at a meeting of the American Mathematical Society, though he has not yet published

them. Further references bearing on the subject will be found at the end of this thesis.

In this thesis I shall, in Chapter II, give the constructions devised by Fossfeld, while in Chapter III, I give the work as represented by Ruth Before, however, proceeding to a consideration of Fossfeld's work, it will be necessary to give some explanation of the manner of representing imaginary elements.

The question of locating imaginary points in the real-plane, in such a manner that they may be used for synthetic construction, may at first sight, seem incapable of solution. This is, however, not the case, for the method given by Von Staudt in his, "Geometrie der Lage", allows us to solve our problem. This method has been repeatedly explained, as in Reye', "Geometrie der Lage." Good representations in English are found in Miss C. A. Scott's, "Modern Analytical Geometry", § 151-152,

and in A. Emch's "Introduction to Projective Geometry and its Applications", §§ 11–51. In the following I briefly outline the method.

When there is a one to one relation between the points of a line, that is when to every point A, of the line, corresponds some other point A', of the line, and conversely A' corresponds to A, then the points are said to form an involution. It may happen that A' and A coincide, in which case this point is called a double point of the involution. It will be shown below that every involution has two such double points. The fact that the point A' corresponds to A may be expressed analytically by the following equation

$$\alpha x x' + \beta x + \gamma x' + \delta = 0 \qquad (1)$$

where x and x' are the parameters of the two systems and the same origin is adopted for both. This equation, however, does not express the fact that, conversely, A corresponds to A'. In order that this be so we must specialize the equation, and

must have the added condition that

$$\frac{\beta\lambda + \delta}{\alpha\lambda + \gamma} = \frac{\gamma\lambda + \delta}{\alpha\lambda + \beta}$$

for all values of λ; this requires that $\beta = \gamma$, and hence (1) becomes

$$\alpha x x' + \beta(x + x') + \delta = 0 \qquad (2)$$

If we wish to find the double points of the involution, we must put $x = x'$; this reduces equation (2) to

$$\alpha x^2 + \beta x + \delta = 0 \qquad (3)$$

which shows that there are two double elements, real or imaginary. In the case of real points, the point midway between the double points is real. By taking this point as the origin, $\beta = 0$ and the involution is expressed by

$$\alpha x x' + \delta = 0$$

that is by

$$x x' = k$$

and the double points are given by

$$X^2 = K$$

Three cases arise, $K > 0$, $K = 0$, $K < 0$. In the first case the double points are real and distinct, and the involution is hyperbolic; in the second case the double points are real and coincident, and the involution is parabolic; in the third case the double points are imaginary, and the involution is elliptic. We have a similar state of affairs, when in place of points on a line we are dealing with rays through a point. Thus we may have an involution of rays which is hyperbolic, parabolic, or elliptic, according as the double rays are real and distinct, real and coincident, or imaginary.

It is the elliptic involution which enables us to fix a pair of imaginary points or lines. We can establish an elliptic involution, of points or rays, either by

giving the algebraic transformation or by giving two pairs of real correspondents of the involution. Then the double elements of this involution are a pair of definite imaginary elements and are thus given. This method of locating imaginary elements is used throughout the following pages.

Since five elements must be given in order to construct a conic, and since some of these elements may now be imaginary, we can readily see that we may have twenty possible cases. These are, however related to each other in such a way that ten are the dualistic of the other ten. These cases are given below in tabulated form.

	Real points	Pairs of imaginary points	Real tangents	Pairs of imaginary tangents
1	5	—	—	—
2	3	1	—	—
3	1	2	—	—
4	4	—	1	—
5	2	1	1	—
6	—	2	1	—
7	3	—	2	—
8	1	1	2	—
9	3	—	—	1
10	1	1	—	1

The other ten cases are the dualistic of those given above.

Chapter II.

Fossfeld's Method.

In this chapter Fossfeld's construction of conics is to be discussed. The different cases are arranged in groups of two, which are dualistic each to the other.

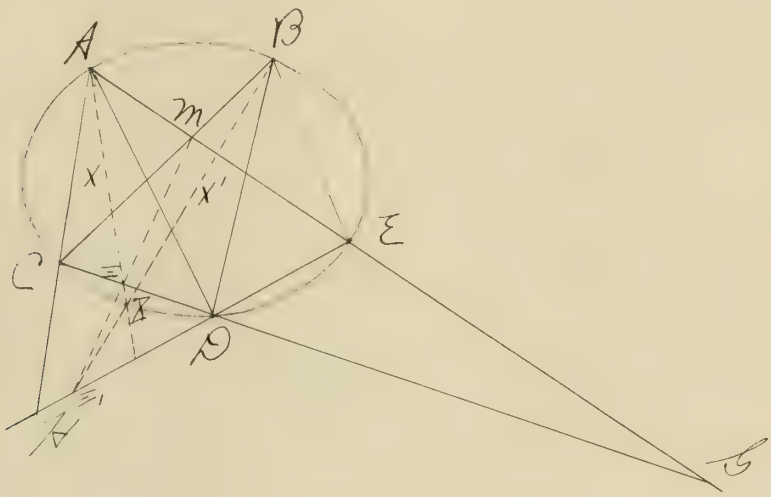
Problem I.

To construct a conic passing through five real points, no three of which are on the same straight line.

Given: The five real points A, B, C, D, E, no three lying on the same straight line.

Construction:- Connect A and B, each, with C, D, and E. Connect D with

C and E. From A draw any ray x, until it meets CD in Ξ. Join M, the intersection of AE and BC, with Ξ. Let MΞ meet DE at Ξ₁, and connect Ξ₁ with B. Now BΞ₁ and AΞ intersect at the point X. Then X is a point of the desired conic.



Proof: Let CD meet AE at G, and let DE meet BC at H. Then the point-rows HΞ, DE and CΞ DG are perspective, and hence projective, M being the center of perspection. Hence the pencils A – CΞ DG and B – HΞ, DE are projective. But X is the

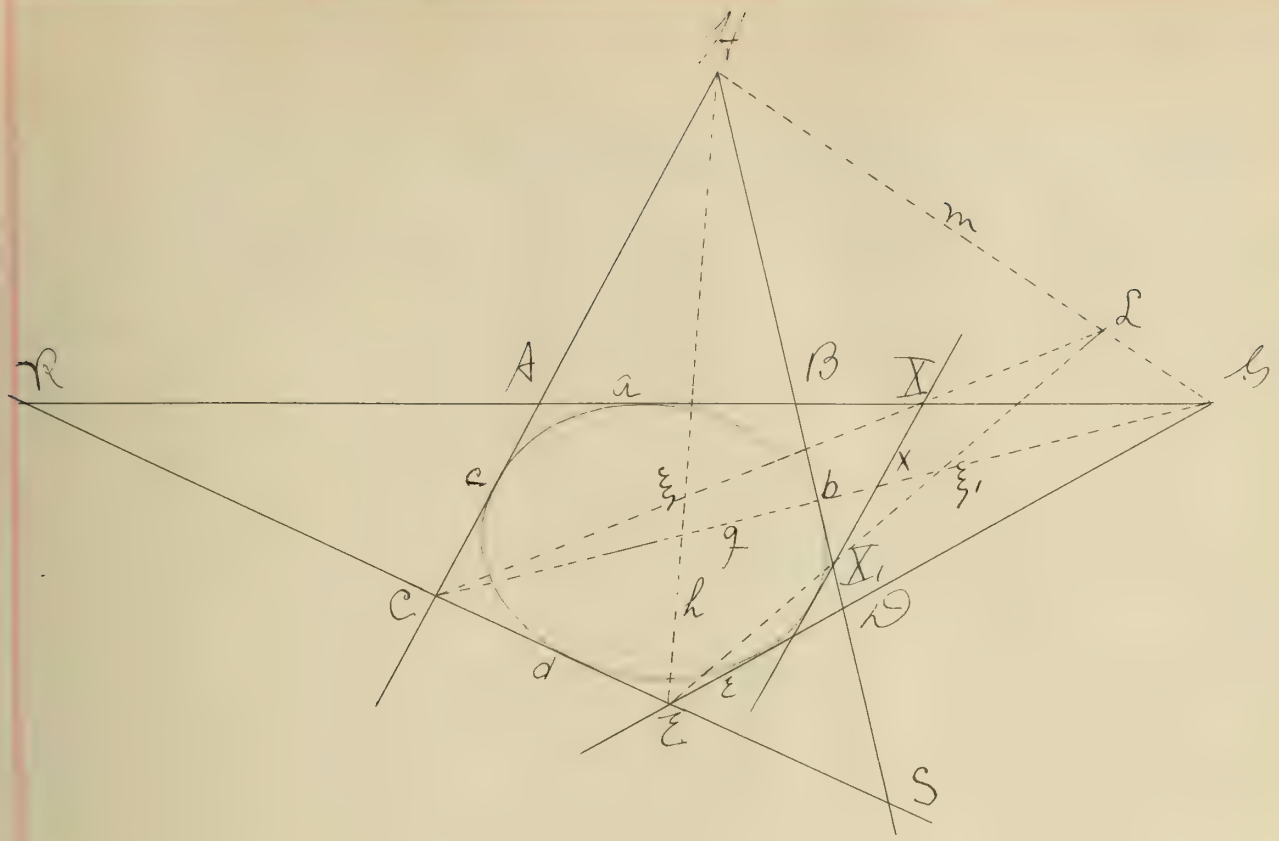
intersection of two homologous rays, and hence is a point of the conic.

Problem Ia.

To construct a conic tangent to five real lines, no three of the lines passing through the same point.

Given: The five real lines a, b, c, d, e,

Construction: Let c, b, and e meet a at A, B, and G, respectively. Let c and d intersect at C, and let d and e intersect at E, and c and b at H, b and e at D. Choose any point I on a, join I with C by the line ξ. Connect the point H with G by the line m. Let the line ξ meet m at L. Join L with E by means of the line ξ_1. Let ξ_1 intersect b at I_1. Join I_1 with I, this will be a tangent to the required conic.

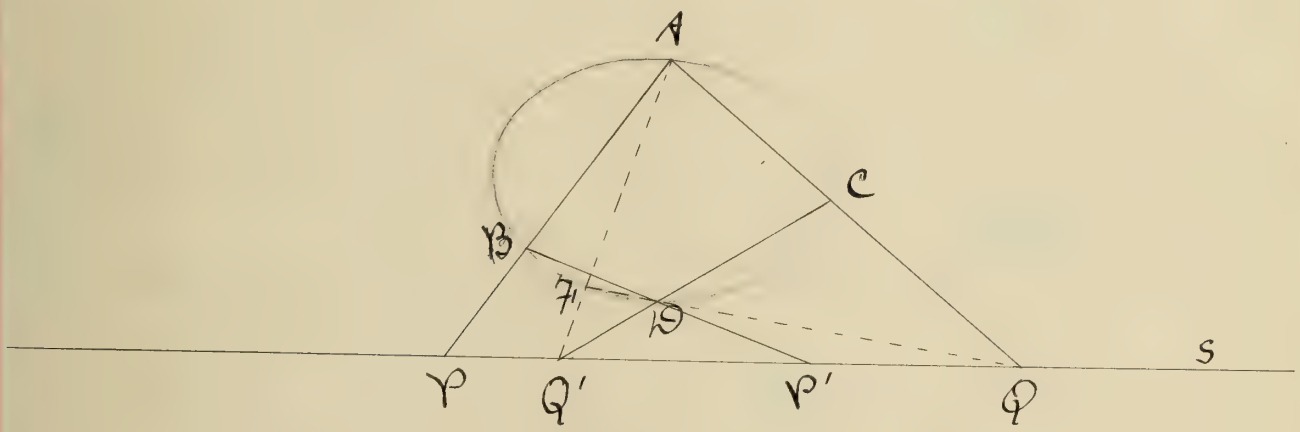


Proof: Join E with K by the line g, and join E with H by the line h. Then the two pencils h ξ, d ε and c ξ, b g are perspective and hence projective, when m is taken as the line from which the two pencils are projected. Hence the points rows A X K and B X, D are projective. But x is the line joining two homologous points, and hence is a tangent to the conic.

Problem 2.

To construct a conic which passes through three real points and two imaginary points.

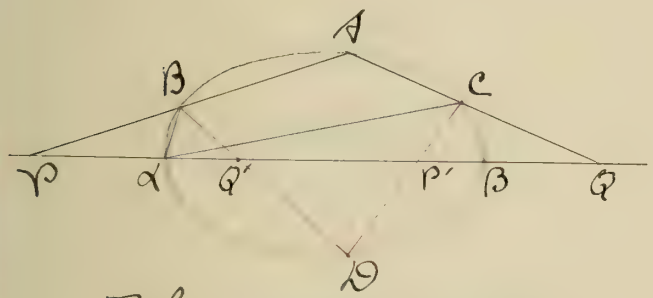
Given. The three real points A, B, C, and the two imaginary points α and β which are the fixed points of an elliptic point involution on s.



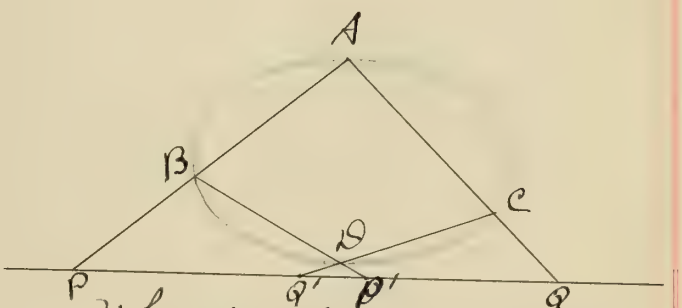
Construction. Connect A and B and let AB meet s at P. Connect A and α and let $A\alpha$ meet s at Q. Join A with P'; the corresponding of Q, with respect to the given involution, and join B with P', the cor-

respondent of P. The point of intersection, D, of BP' and CQ' is a point on the desired conic. Join A with Q' and D with Q. The intersection, F, of AQ' and DQ is also a point on the conic. We now have five real points of the required conic.

Proof. We must first prove that D is a point on the conic.



When α and β are real.

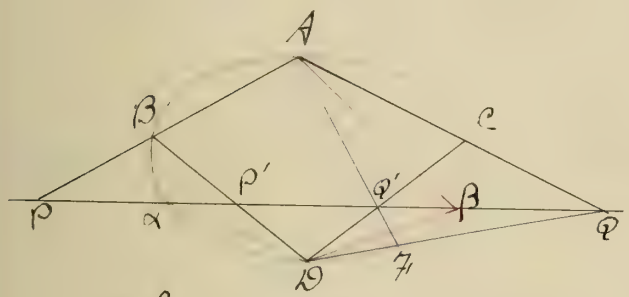


When α and β are imaginary.

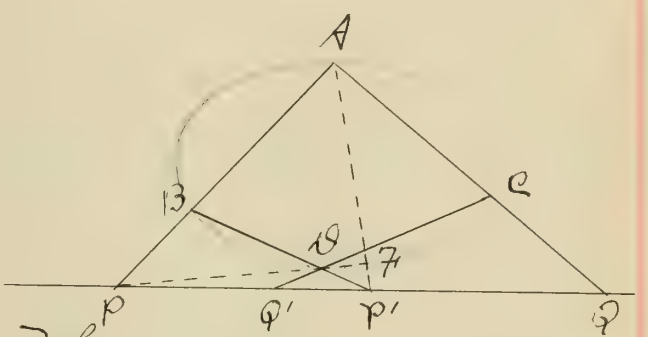
The pencil of rays B-PαP'β and C-αQ'βQ, at B and C are harmonic, because the ranges of points PαP'β and αQ'βQ are harmonic ranges. Since the locus of the intersection of homologous rays of two homographic pencils is a conic through the vertices of the pencils, the homologous rays Cα and

$B\alpha$, BP and CQ, and $B\beta$ and $C\beta$ intersect on the conic. Therefore BP' and CQ' must be homologous. Now BP' and CQ' intersect at D. Therefore D is a point of the required conic

We must next prove that γ is a point on the conic.



When α and β are real



When α and β are imaginary.

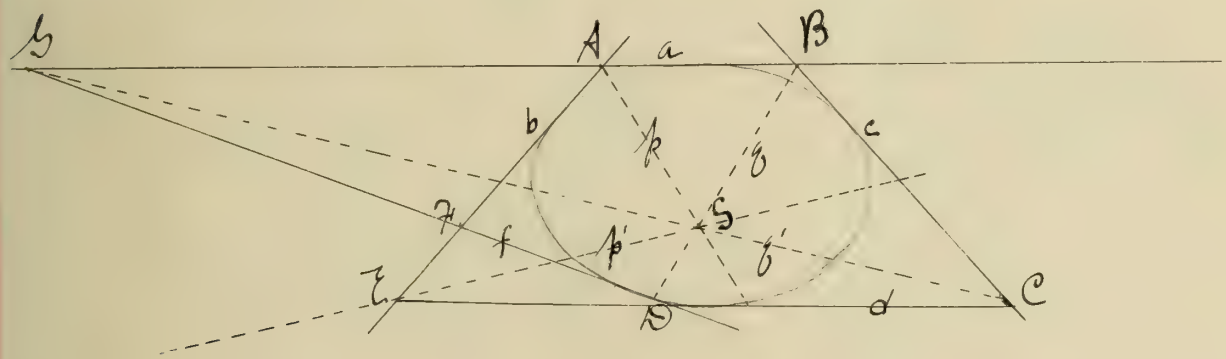
The pencil of rays $A-\alpha Q'\beta Q$ and $D-\alpha Q'\beta Q$ at A and D are harmonic, since the range of points $\alpha Q'\beta Q$ is a harmonic range. Since the locus of the intersection of homologous rays of two homographic pencils is a conic through the vertices of the pencils, the homologous rays $A\alpha$ and $D\alpha$, AQ and DQ', $A\beta$ and $D\beta$ intersect on the conic. Therefore AQ' and DQ are homologous rays. Now AQ' and DQ in-

tersect at F. Therefore F is a point on the desired conic.

Problem 2 a.

To construct a conic tangent to three real lines and two imaginary lines.

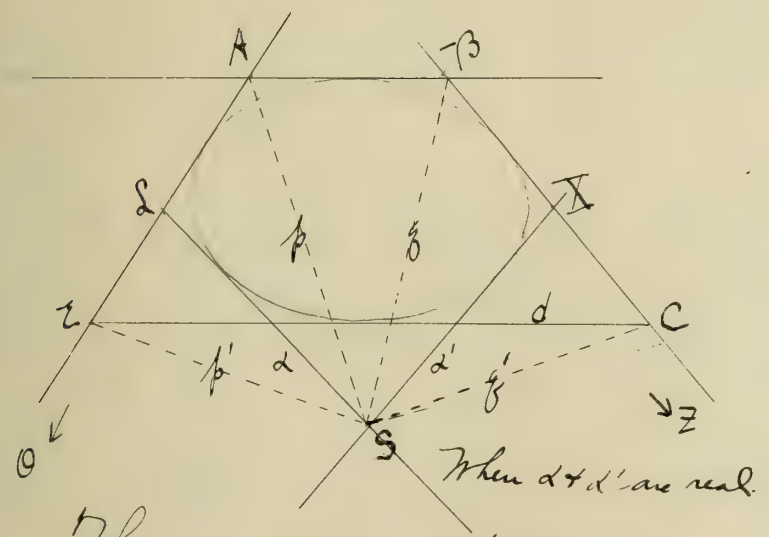
Given: The real lines a, b, c and the imaginary lines α and α', which are the fixed rays of an elliptic involution at S.



Construction. Let a and b intersect at A, and b and c at B. Join A and B with S by the rays p and g. Find their correspondents, p' and g', with respect to the given involution. Let p'

meet b at E, and g' meet c at C. Join E and C. Then EC is a tangent to the conic. Let g' meet a at G and let g meet EC at D. Join G with D. Then GD is also a tangent to the required conic. We have now obtained five real tangents to the desired conic.

Proof. We must first prove that EC is a tangent of the conic.

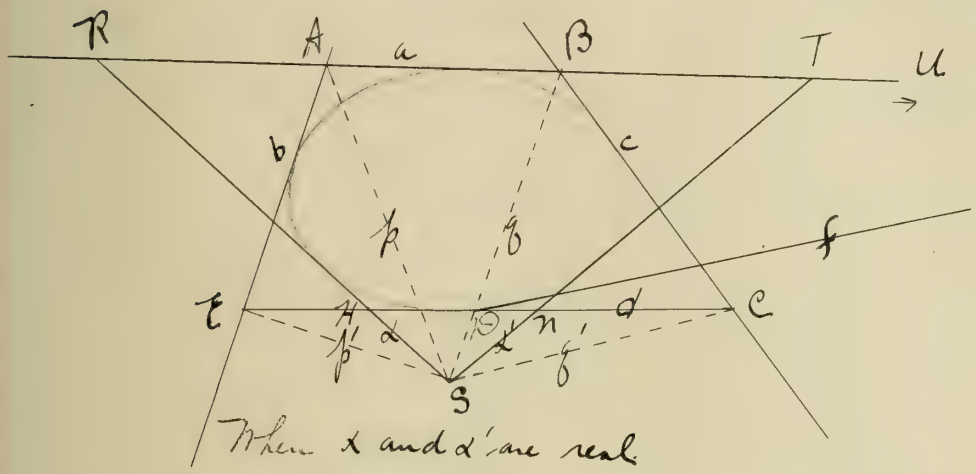


When α and α' are imaginary figure is same as used for construction.

The range of points A, L, E, O, in a and the range B, X, C, Z in c are harmonic. The lines homologous points are tangent to the conic. A and B, L and Z, O and X are homologous. Therefore E and C must be homo-

logous and the line E C joining them is tangent to the required conic.

We must now prove that G D is a tangent of the conic.



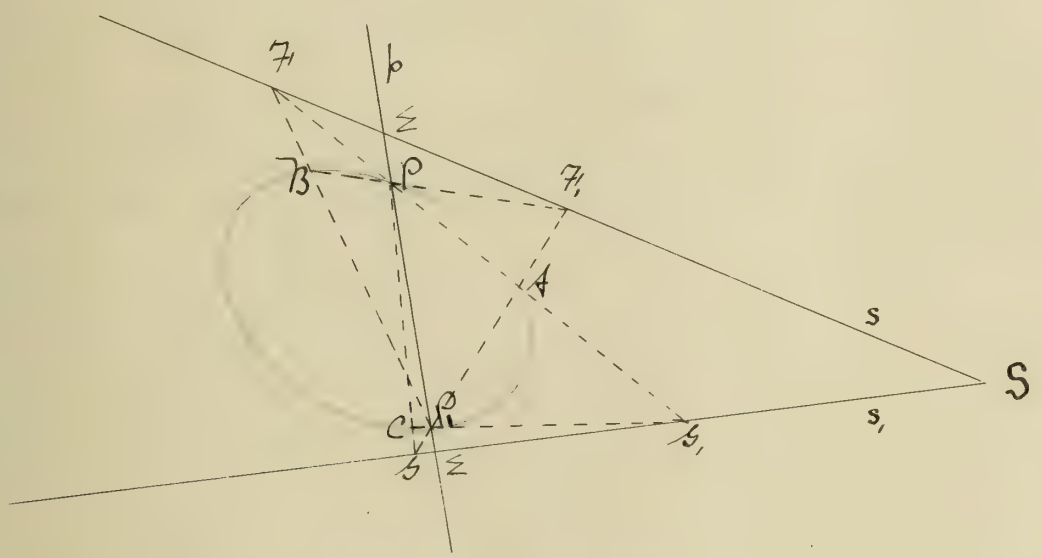
When α and α' are imaginary the figure is same as that used for construction

The ranges of points R, B, T, U and H, D, N, C, on a and E C are harmonic. The lines joining homologous points are tangent to the conic. R and H, B and C, T and N are homologous, therefore D and U must be homologous, and the line D G, joining them, must be tangent to required conic.

Problem 3.

To construct a conic which passes through one real and two pair of imaginary points.

Given. The real point A and the imaginary points α and α' which are the fixed points of an elliptic involution on s, and β and β' which are the fixed points of an elliptic involution on s'



Construction. Let s_1 meet s at S. The correspondent with respect to the given involution, of S on s is Σ, and on s' is Σ_1. Join Σ and Σ_1, by the line p. The given point involution on s, determines an involution of rays through A. The

given point involution on s determines an involution of rays through A. The common pair of rays of these two involutions meet s in F and F_1, and meet s_1 in G and G_1. Therefore F and F_1 are conjugate and G and G_1 are conjugate. AF and AG_1 meet p in P and P_1, which are points of the required conic. FP and F_1P_1 meet at B, which is a point of the required conic. G_1P and GP_1 meet at C, which is also a point of the required conic. We have thus found five real points of the conic and the conic is therefore determined by the construction of problem I.

Proof. $\Sigma\Sigma_1$ is the polar of S, by construction GF, F_1G_1, are the common elements of two involutions, therefore F and F_1, and G and G_1 are conjugate. From problem 2, FA must meet the polar of S on the desired conic, therefore FA must meet p on the conic. That B and C are points of the conic follows from the proof of problem 2.

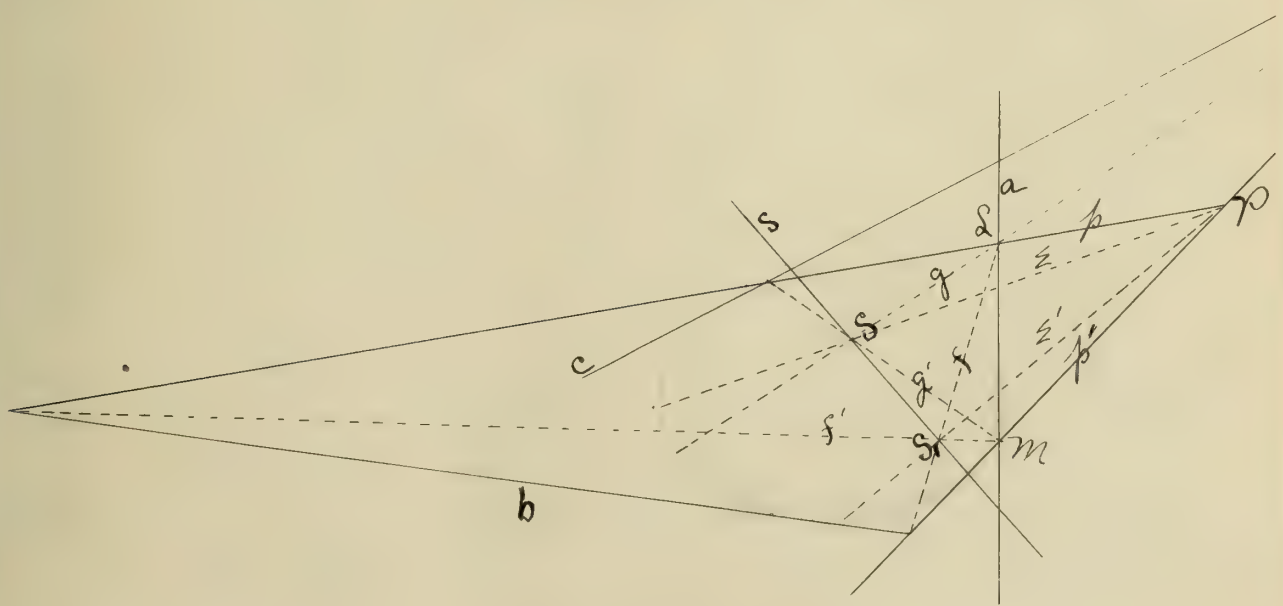
Problem 3a.

To construct a conic tangent to one real line and to two pairs of imaginary lines.

Given. The real line a, and the imaginary lines α and α', which are the fixed rays of an elliptic involution at S, and the imaginary lines β and β' which are the fixed rays of an elliptic involution at S_1.

Construction. Join S and S_1 by means of the line s. Let Σ be the ray corresponding to s at S and Σ_1 the ray corresponding to s at S_1, with respect to the given involutions. The involutions of rays at S and S_1 determine two point involutions ~~at~~ on a. The common pair of points of these two involutions determines the rays g and g_1 at S, and f and f_1 at S. Therefore g and g_1, and f and f_1, are correspondents. The line p, joining Σ, the intersection of a and f, with P, the intersection of Σ and Σ_1, is a tangent of the desired conic

as also is the line p_1, joining M, the intersection of a and g, with P. The line b, joining the intersection of f_1 and p_1, with the intersection of f_2 and s_1, is a tangent to the conic. The line c, joining the intersection of g_1 and p_1, with the intersection of g_2 and p_2, is also a tangent to the conic. We have now found five real tangents to the desired conic, and hence it is determined by the construction of Problem I a.



Proof. The proof is the exact dualistic of problem 3.

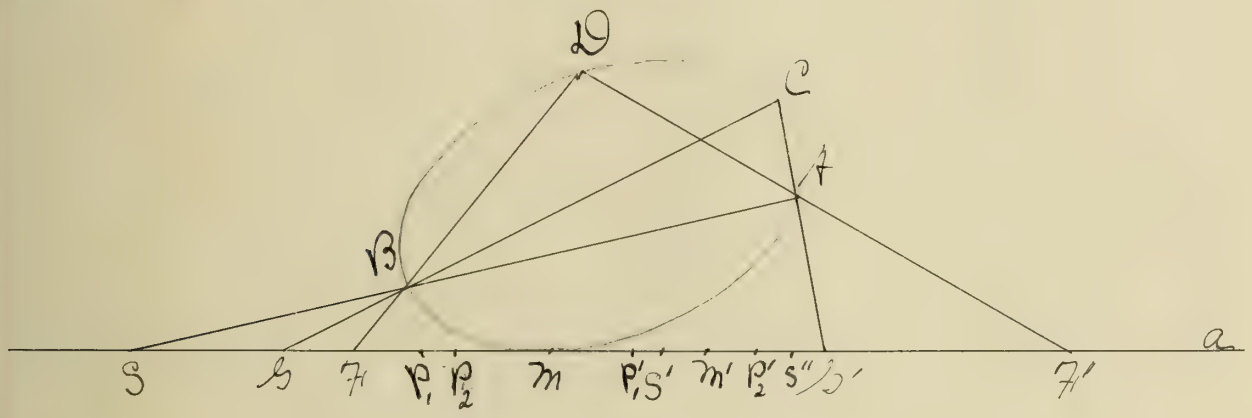
Problem 4.

To construct a conic passing through four real points and tangent to one real line, which does not contain either of the given points.

Given: The real points A, B, C, and D, and the real line a.

Construction. Join D with B and A, and let DB meet a in F, and DA meet a in F'. Join C with B and A, and let CB meet a in G, and CA meet a in G'. Let G and G' and F and F' be correspondents in two projective point ranges on a. Join A with B, and let AB meet a in S. Let any other point, S', on a be the correspondent of S in the point ranges. We now have the two projective point ranges S, G, F and S', G', F'. Find the double points P_1 and P_1' of these point ranges. [See Reye, "Geometrie der Lage".]. Choose any other point S'' be the correspondent of S in the range. Find the double

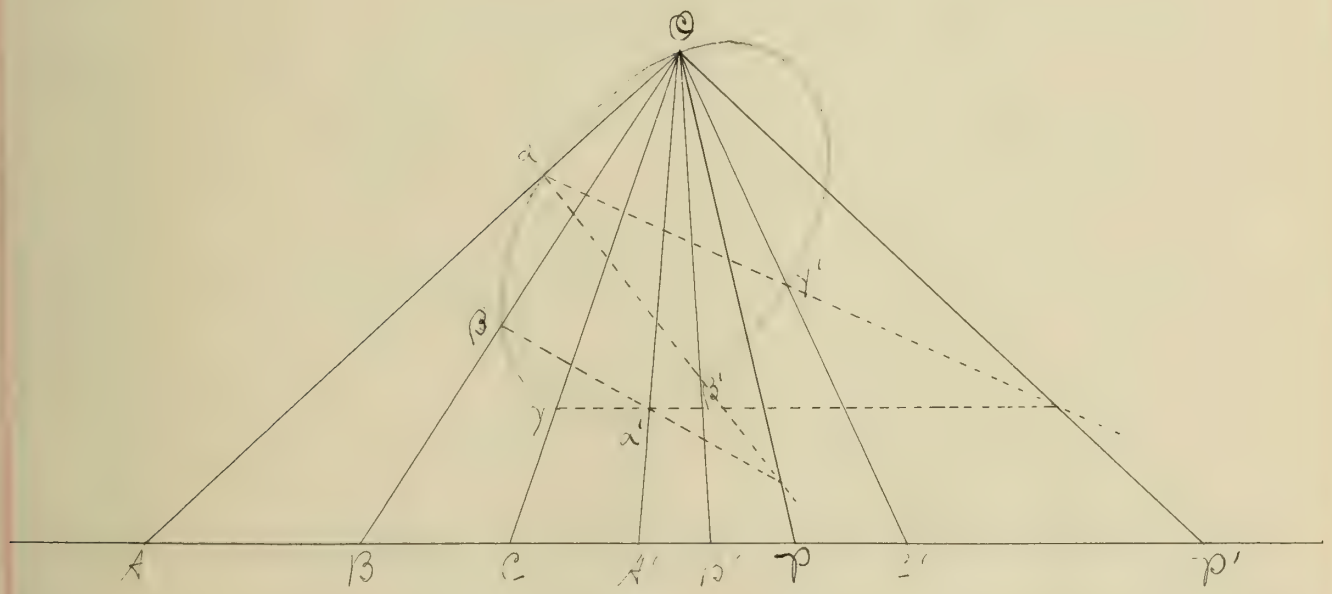
points, P_2 and P_2', of the new ranges, S, G, T, and S", G', T'. The points P_1 and P_1', and P_2 and P_2' then determine an involution on a. Find the double points of this involution. [See C. A. Scott, "Modern Analytical Geometry," p 163]. The double points, M and N, of the involution will be the points of tangency of the conic with the line a. There will be two real solutions since there are two double points of the involution.



Note. To find the double points of two projective point ranges.

Given. The two projective point ranges, A, B, C, and A', B', C'.

Construction. Join any point O, to any conic with A, B, C, A', B', C'. Let OA cut the conic at α, OB at β, OC at γ, OA' at α', OB' at β', OC' at γ'. Join α with γ' and α' with γ. Let $\alpha\gamma'$ and $\alpha'\gamma$ meet at N. The point of intersection, P, of ON with the line of the double points. Join β with α' and α with β'. Let $\beta\alpha'$ and $\alpha\beta'$ meet at M. The point of intersection, P', of OM with s is the second double point

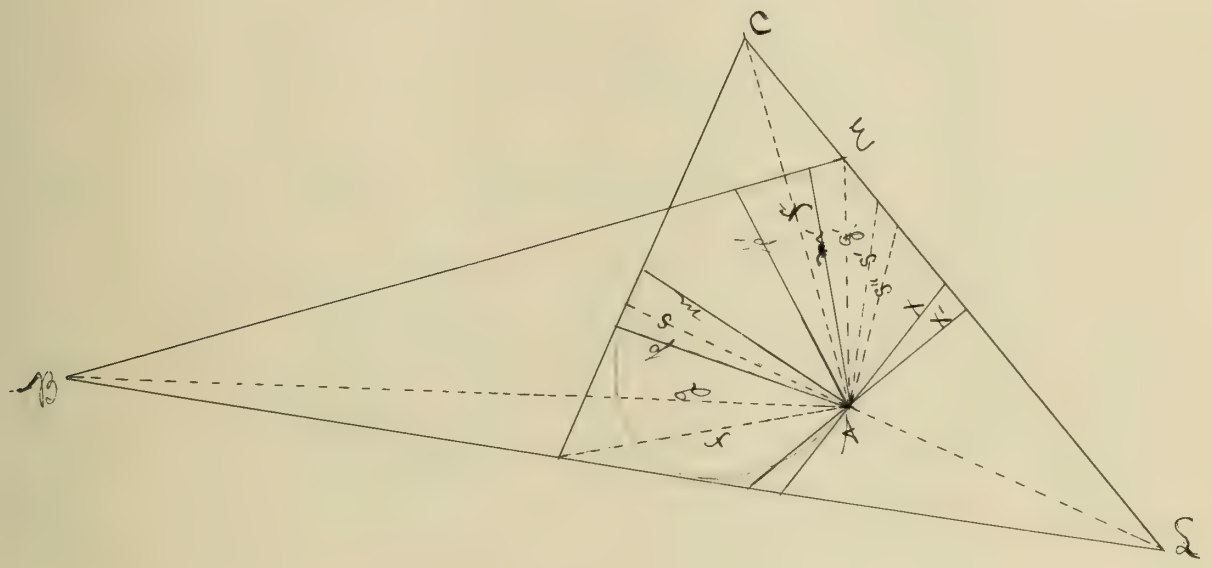


Prop. The four points A, B, C, D, determine a family of conics. These conics cut the line a in points which form an involution. The double points of the projective point ranges also form an involution on a. It is required to show that these two involutions are identical. A and B may be considered as the vertices of two pencils ranges. Corresponding rays intersect on the conic. Now D, the intersection of GD and G'D is a point of the conic; C the intersection of GC and G'C is a point of the conic; A the intersection of SA and S'A is also a point of the conic. P_1 and P_1' are common to the two point ranges, therefore they must correspond to themselves, hence BP and PA, and BP' and P'A intersect on the conic. It can be proved in the same way that the conic which intersects at P_2, also intersects at P_2'. The conics which pass through the double points of this involution are the desired conics, since then they have two coincident points in common with the line.

Problem 46.

To construct a conic tangent to four real lines and passing through one real point.

Given. The four real lines a, b, c, d, and the real point A.



Construction. Let d meet b at D and meet a at C. Join D and C with A by means of the lines f and f'. Let c meet b at B and meet a at E. Join B and E with A by means of the lines g and g'. Let f and f' and g and g' be corresponding rays of a pencil range at A. Let a and b meet at L. Join L with

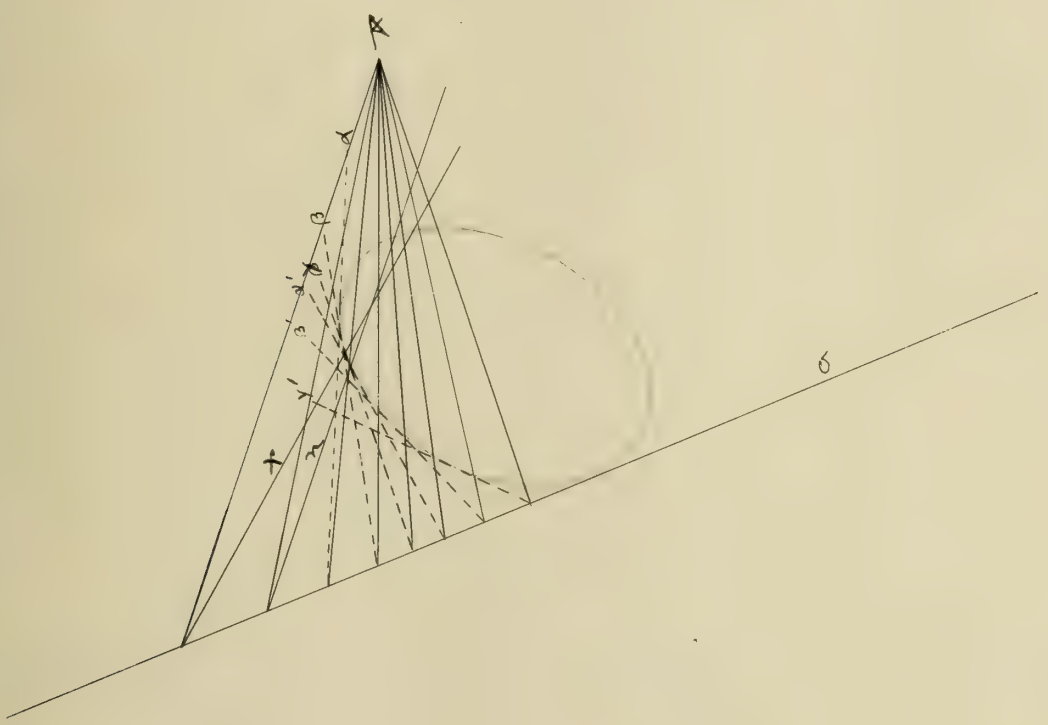
A by means of the line s. Draw any other ray s' and let it be the correspondent of s in the pencil range at A. We have now two projective pencil ranges at A, s, f, g, and s', f', g'. Find the double rays p and p'. Let any other ray s'' be the correspondent of s. Find the double rays, m and m', of the new pencil ranges, s, f, g, and s'', f', g'. The rays p and p', and m and m' then determine an involution of rays at A. Find the double rays x and x' of this involution and they are the rays through A tangent to the required conic. Since there are two double rays of the involution there will be two solutions, both real.

Note. To find the double rays of two projective pencil ranges.

Given. The rays a, b, c, a', b', c' through A.

Construction. Draw any tangent to any conic. Find the intersections of the rays with the tan-

gent, σ. From these points draw the tangents α, β, γ, α', β', γ'. Find the intersection of α and γ', and of α' and γ, join these points by the line π. Join the intersection of π and σ, with A, this is a double rays. Find the intersection of α and β' and of α' and β, join these points by means of the rays χ. Join the intersection of χ and σ, with A, this is a second double ray.

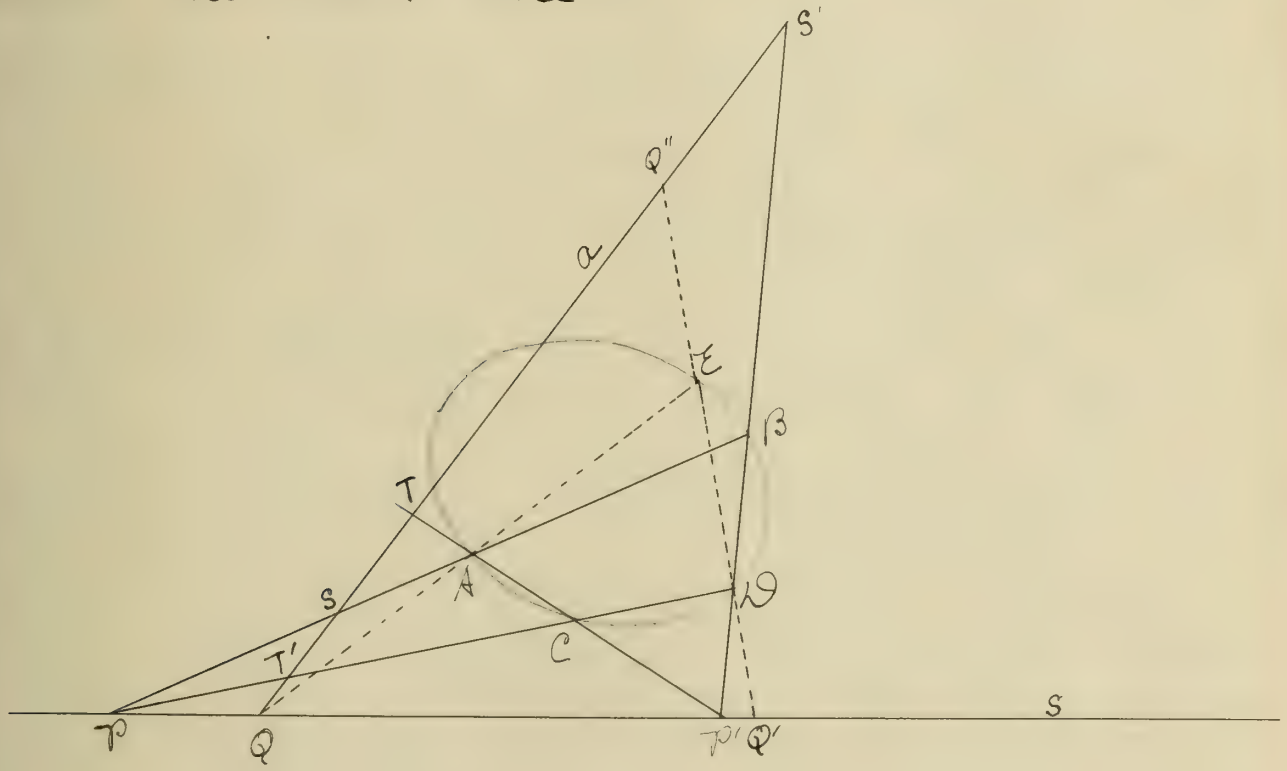


Proof. The proof is the exact dualistic of problem 4.

Problem 5.

To construct a conic passing through two real and two imaginary points and tangent to a real line.

Given. The real points A and B, and the imaginary points α and α', which are the fixed points of an elliptic involution on s, and the real line a.



Construction. Let $\alpha\beta$ meet s in P, and let P' be the correspondent of P with respect to the given involution on s. Connect P' with A and B.

Through P draw any line cutting P'A in C and P'B in D. Let a meet s in Q, and let Q' be the correspondent of Q with respect to the given involution on s. Join Q with A and Q' with D, and let QA meet Q'D in E. Then the points A, B, C, D, E, determine a conic through the two given imaginary points. By rotating PD about P, we can construct a whole family of conics, and we must find that conic of the family which is tangent to the line a. PA cuts a at S, P'B cuts it at S', P'A cuts it at T, PD cuts it at T', and Q'E cuts it at Q". Of these points Q, T, S, and S' are fixed, while T' and Q" vary as PD varies. Take one position of PD and find the double points of the projective point ranges Q, S, T, and Q", S', T'. Let these be F and F'. Take a second position of PD and find the double points G and G' of the two resulting point ranges. FF' and GG' determine an involution, whose double points are the points of tangency of the desired conic

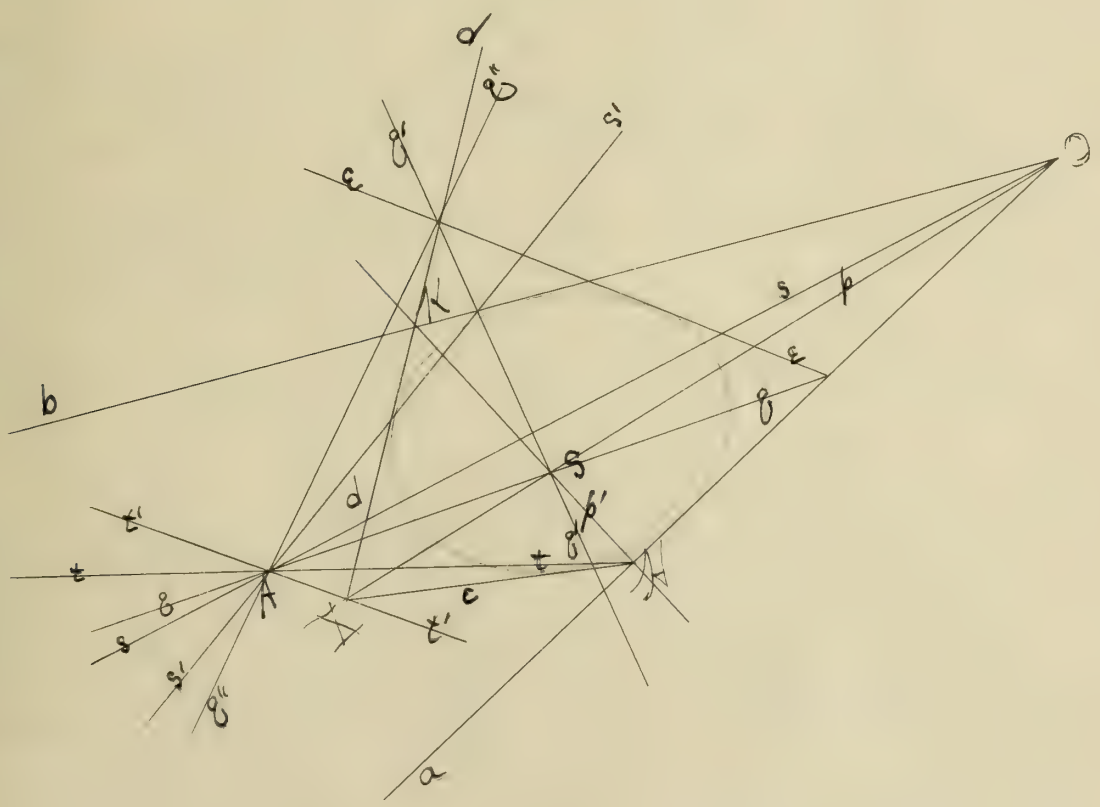
with a. There ~~with~~ are two real solutions if the involution on a is hyperbolic, and imaginary if the involution is elliptic.

Proof. Through the two real and two imaginary points a family of conics can be drawn. Since P' is the conjugate of P, it follows that P'A and P'B must cut any one of these conics in points which are collinear with P. And conversely, if any line through P cuts P'A and P'B in C and D respectively, then C and D are points on one of this family of conics. [Reye, "Geometrie der Lage", p. 139]. We must now prove that E is a point of this conic. A and D may be considered as the vertices of two pencils of rays. Since Q and Q' are conjugate points, the rays QA and Q'D are conjugate rays, of the two pencils, and hence their intersection, E, is a point of the conic. It follows from the proof of problem 4 that the construction for the points of tangency with a is correct.

Problem 5a.

To construct a conic passing through one real point and tangent to two real and two imaginary lines.

Given. The real point A, and the real lines a and b, and the two imaginary lines c and c', which are the fixed rays of an elliptic involution at S.



Construction. Join the intersection O of a and b, with S by means of the line f, and find the correspondent f' of f, with respect to the given involution. Let f' meet a and b in the points A' and K respectively. Choose any point, I, on f and connect it with A' and K' by means of the lines c and d. Join A with S by the line g, and find g' the correspondent of g, with respect to the given involution at S. Join the intersection of g and a with the intersection of g' and d, by the line e. Then the lines a, b, c, d, e, determine a conic through the two given imaginary rays. By letting the point I move along f, we can construct a whole family of conics, and we must find that conic of the family which passes through A. Join A with O by the line s, with the intersection of f' and b by the line s', with the intersection of f' and a by the t, with the intersection of f and d by the line t, and with the intersection of g' and e by the line g''. Of these rays, s, t, s and s' are fixed, while t' and g'' vary

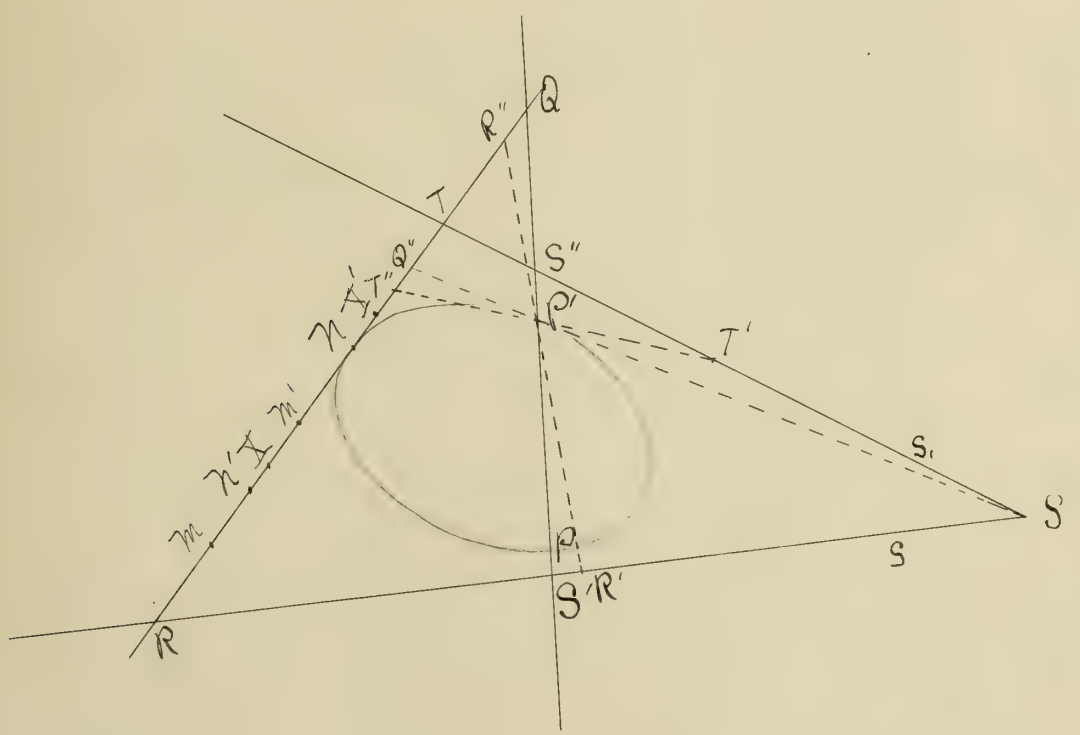
as X moves along p. Take one position of X and find the double rays of the projective pencil ranges s, s, t, and s'', s', t'. Let these be f and f'. Take a second position of X and find the double ~~points~~ rays x and x' of the two resulting pencil ranges. ff' and gg' determine an involution, whose double rays are the tangents to the conic.

Proof. The proof is the exact dualistic of problem 5.

Problem 6.

To construct a conic passing through two pair of imaginary points and tangent to a given real line.

Given: The line a, and the imaginary points δ and δ', which are the fixed points of an elliptic involution on S, and β and β', which are the fixed points of an elliptic involution on S_1.



Construction. Find the point S'' which corresponds to S on s_1, and the point S' which corresponds to S on s. Let T and R be the intersections of a with s_1 and s. Let T' and R' be the correspondents of T and R, respectively on s_1 and s. The line p, joining S'' and S', is the polar of S. Choose any point U_1 and construct a conic through U_1, $\alpha, \alpha'; \beta, \beta'$ by means of problem I. This conic will cut the line p in the points P and P'. Join P' with R', S, and T and let R'P' meet a at R''; P'S at Q''; P'T at T''. Call the intersection of a and p, Q. From problem 4 the intersections, M and M', of the conic with a are the double points of the pair of projective point ranges found by Q, T, R, and Q'', T'', R''. Choose any other point U_2. Use the same construction to find the intersections X and X' of the conic passing through U_2, $\alpha, \alpha'; \beta, \beta'$, with a. By choosing U in different positions the intersections of the conics with a form an involution. The double points N and N' of this in-

are the points of tangency with the desired conics. By problem 3 construct through each of these points and the imaginary points. There will be two solutions.

Proof. By problem 3 the points P and P' are the intersections of the conic through U, α, α', β, β' with s. By problem 4 the points N and N' are the points of tangency with the line a.

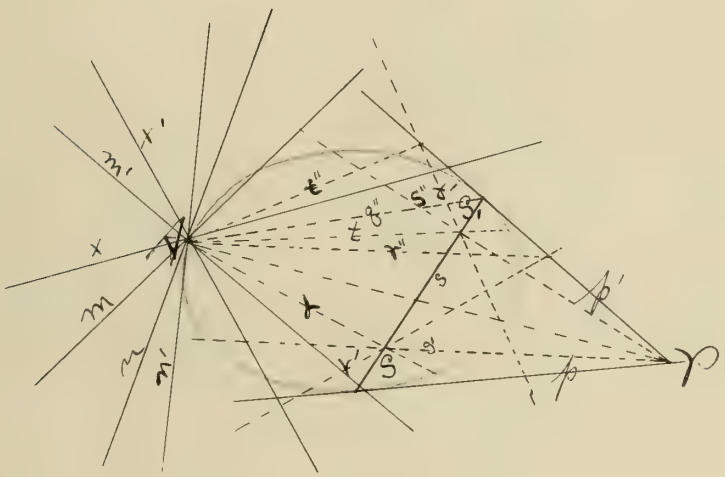
Problem 6a.

To construct a conic passing through one real point and tangent to two pair of imaginary lines.

Given. The real point A, and the imaginary lines α and α' and β and β', which are the fixed rays of elliptic involutions at S_2 and S_1.

Construction. Join S and S_1 by means of the line s, let s' and s'' be the correspondents of s with respect to the given involutions at S and S_1. Join A with S by means of the line t, and with S_1 by means of the line t. Find the correspondents t' and t' of t and t. If the intersection of s' and s'' is the pole of s. Choose any line u, and construct a conic tangent to $u, \alpha, \alpha'; \beta, \beta'$ by the construction used in problem 3a. This conic is tangent to the lines s and s'' through P. Find the intersection of s'' with t', S and t', and join the points of intersection with the point A by means of the lines t'', S'', and t''. Join A and P by the line g. From problem 4a the tangents u and u' through A, of the conic are the double rays of the projective pencil ranges formed by t, g, t'' and t', g'', t''. Choose any other line u_2 and use same construction to find the tangents x and x' through A, of the conic tangent to $u_2, \alpha, \alpha'; \beta, \beta'$. By choosing u in different positions the

tangents through A form an involution. The double rays of this involution are the tangents, through A, of the desired conic. By means of problem 3a construct a conic tangent to each of these lines and to the given imaginary lines. There will be two solutions.



Proof. The proof is the exact dualistic of problem 6.

Problem 7.

To construct a conic passing through three real points and tangent to two real lines.

Given. The three real points A, B, C, and the real lines a and b.

Fig. 1

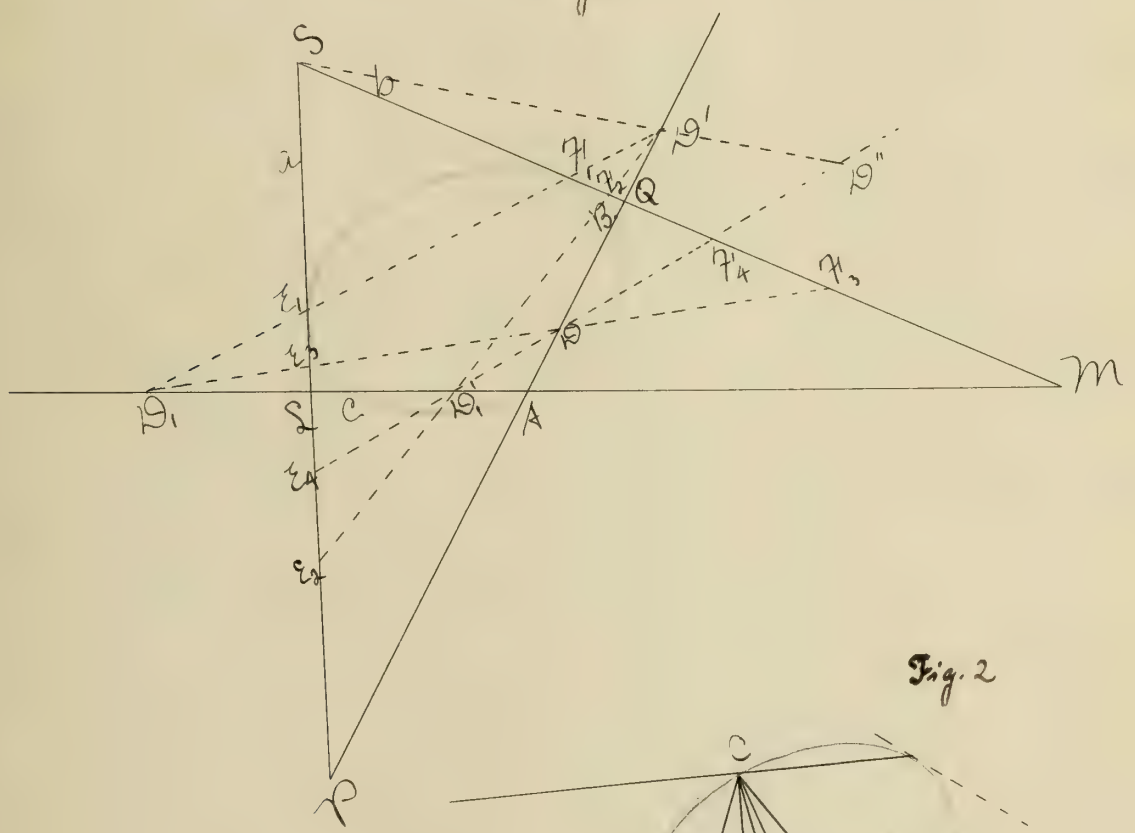
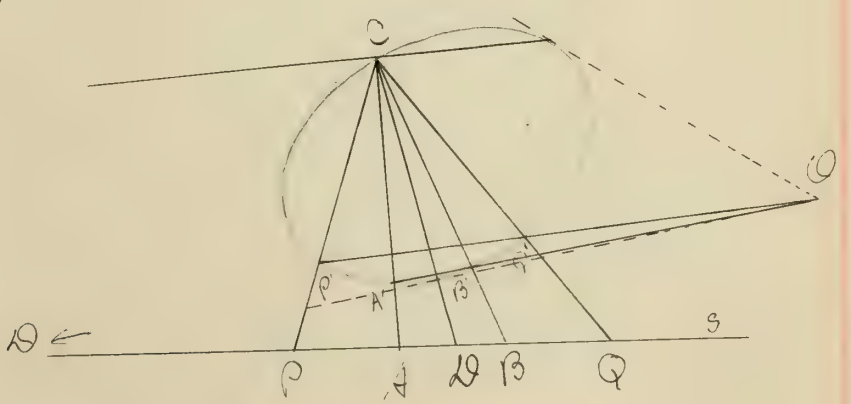


Fig. 2



Construction. Let AB meet b at Q and a at P. Let AC meet b at M and a at L. Find the points D and D' which divide P, Q and A, B harmonically. Find the points, D_1' and D_1, which divide L, M and A, C, harmonically. Join D_1 with D'; ~~and~~ D_1 D' meets a and b in the points E_1 and F_1, which are the points of tangency of a conic satisfying the given conditions. Join D_1' with D; D_1' D meets a and b in E_2 and F_2, which are points of tangency of a conic satisfying the given conditions. Join D_1 with D; D_1 D meets a and b in E_3 and F_3, which are points of tangency of a ~~to~~ conic satisfying the given conditions. Join D_1' with D; D_1' D meets a and b in E_4 and F_4, which are the points of tangency of a conic satisfying the given conditions.

Note. To find the points which divide two pair of points, harmonically. (See fig. 2) Take any conic and using any point of the conic as a center, project the four points onto the conic. Join P' with Q' and

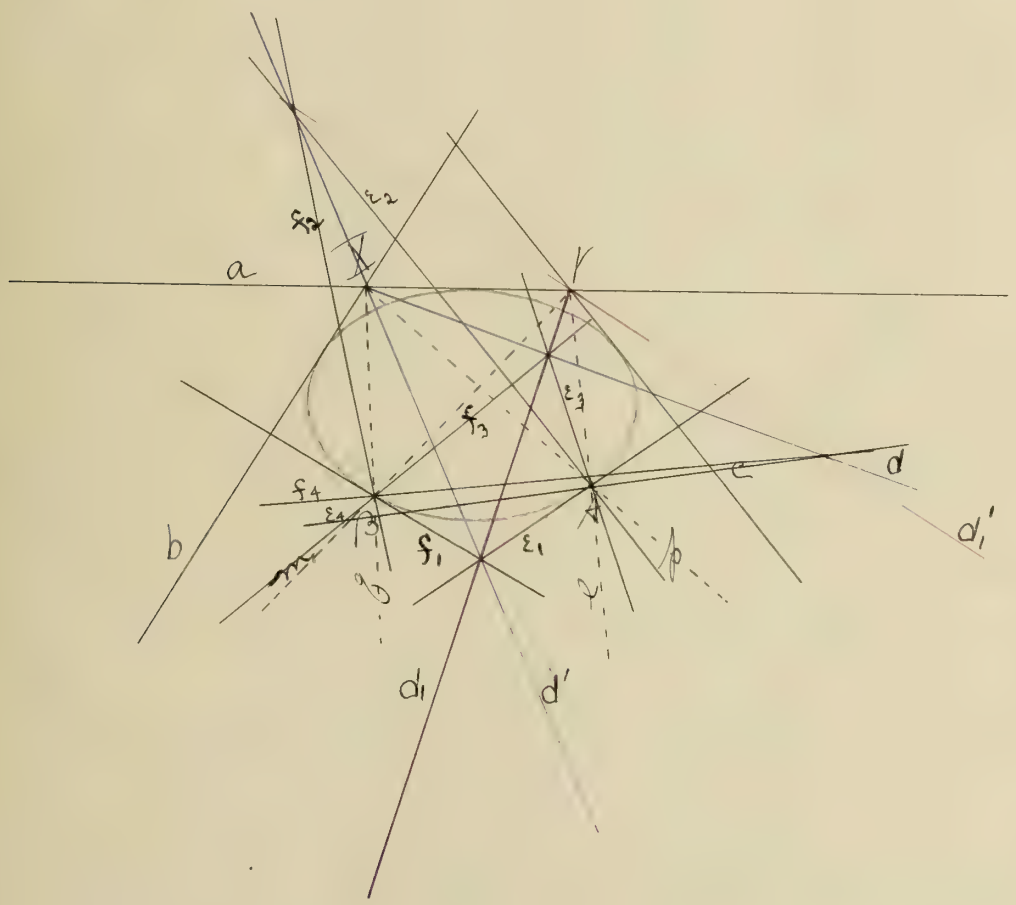
A'B' with B'. Let P'Q' meet A'B' at D. Draw the tangents from to the conic. Join the points of tangency with C, the center of inversion, the intersections of these lines are the required points.

Proof. It is required to prove that E_1 and F_1, E_2 and F_2, E_3 and F_3, E_4 and F_4 are points of tangency of conics with a and b. The points P, D, Q, D' form an harmonic range since D and D' divide P and Q harmonically. Therefore the rays SP, SD, SQ, and SD' form a harmonic pencil. The points E_4, D, F_4, and D'' form a harmonic range since they are the intersections of the harmonic pencil with a straight line. Therefore the line D'' passes through S. It may be proved in the same way that the line D, passes through S. Therefore the points E_4 and F_4 must be the points of tangency with a and b, of one conic satisfying the given conditions.

Problem 7a.

To construct a conic passing through two real points and tangent to three real lines.

Given. The real-points A and B, and the real-lines a, b, c.



Construction. Let a and b intersect at X and a and c at Y. Join X with A and B by means of the lines p and q. Join Y with A and

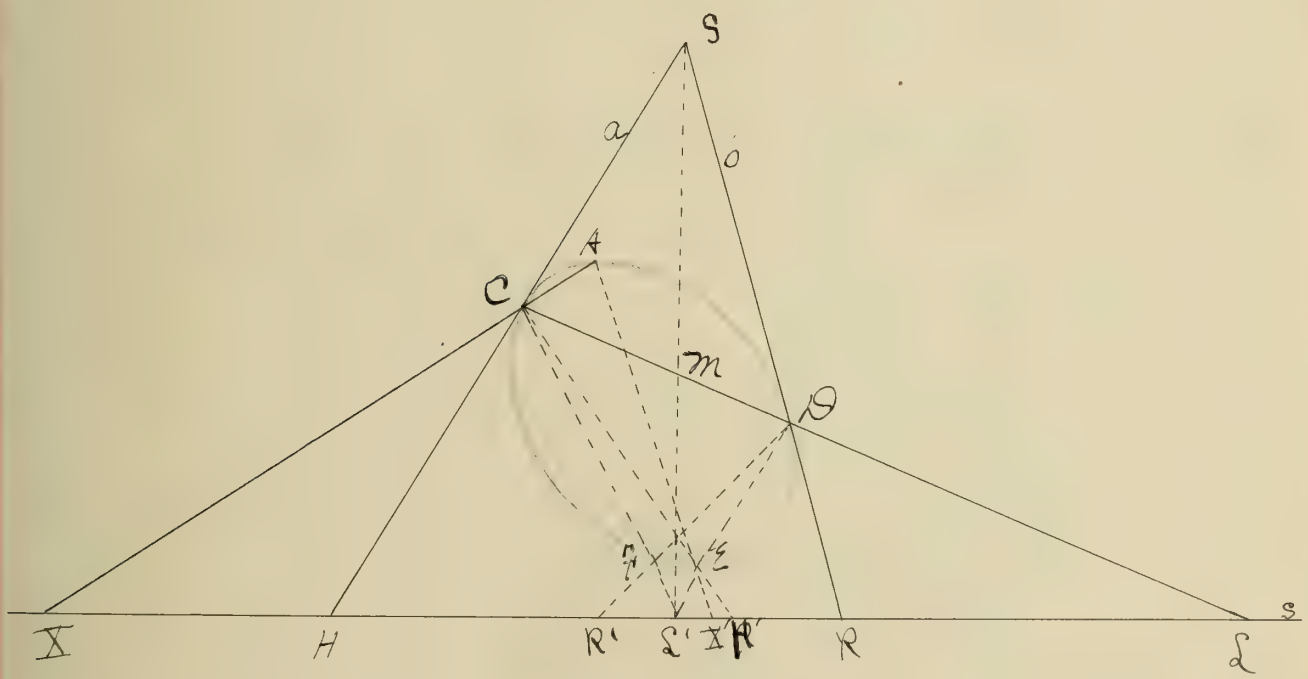
B by means of the lines l and m. Find the rays d and d' which divide f, g and a, b harmonically, also find the rays d_1 and d_1' which divide l, m, and a, c harmonically. Find the intersection of d_1 with d', join this point with A and B by the lines e_1 and f_1, which are tangent to a conic satisfying the given conditions. Join the intersection of d_1' and d' with A and B by the lines e_2 and f_2, which are tangent to a conic satisfying the given conditions. Join the intersection of d_1 and d with A and B, by the lines e_3 and f_3, which are tangent to a conic satisfying the given conditions. Join the intersection of d_1' and d with A and B by the lines e_4 and f_4, which are tangent to a conic satisfying the given conditions.

Proof. The proof is the exact duplicate of problem 7.

Problem 8.

To construct a conic passing through one real and two imaginary points, and tangent to two given real lines.

Given. The real point A and the imaginary points α and α' which are the fixed points of an elliptic involution on s, and the real lines a and b.



Construction. Let a and b intersect at S. Let a meet s at H, and b at meet s at R. Let R' and H' be the correspondents of R and H on s, with re-

sect to the given involution. There is just one pair of points of the involution which divides A and R harmonically. Let L and L' be that pair of points. Through L draw any line meeting a and b at C_1 and D_1. Join R' with D_1 and A' with C_1, then R' D_1 and A' C_1 ~~will~~ intersect at P on S L'. Connect L' with D_1 and A' with C_1, then L' D_1 and A'' C_1 ~~will~~ intersect at E_1. Join D_1 with R' and C_1 with L', then D_1 R' ~~will~~ and C_1 L' intersect at F_1. The line E_1 F_1 passes through L. Connect A with C_1 and E_1, A C_1 and A E_1 cutting s in $\bar{X}_1$ and X_1'. Draw through L a second line, cutting a and b in C_2 and D_2, and repeat the construction. A C_2 and A E_2 determine a second pair of points $\bar{X}_2$ and X_2'. The points X_i, $\bar{X}_i'$ determine two projective point ranges on s, as the line rotates about L. Find the pair of points of the range which coincides with a pair of points of the involution. Call them X and $\bar{X}'$. Join X' with A, ~~and~~ cutting a at C_2, join $\bar{X}'$ with A and C_2 with A', then $\bar{X}'$ A and C_2 A' intersect at E. Join R' with D

and ? with L', then R'D and C L' intersect at F. The five points A, C, D, E, F determine a conic satisfying the given conditions.

Proof. Through the two imaginary points a family of conics can be drawn which are tangent to the two real lines, a and b. The first ~~to~~ to be proved is that any line through L ~~with~~ meets a and b in the points of contact of this family of conics. The points D, C, F, and the two imaginary points determine a conic, and it follows from Problem 2 that the conic also passes through E. Since L and L' divide A and R harmonically the pencil S-A'L'RL is harmonic. The intersection P_1 of A, C_1 and R'D, is the pole of S. Since L lies on s the polar of L passes through P_1 and is L' P_1. Therefore L' P_1 must cut L C_1 at a point M_1, such that C, M_1, D, L is a harmonic range. Also S L' cuts C_1 D_1 in a point M_1', such that C, M_1' D, L is a harmonic range; therefore M_1 and M_1' are one and the same point, and P_1 lies on S L'. Therefore

ΣC_1 is the pair of S and the conic is tangent to a and b at C_1 and D_1. In the same way it may be proved that a second conic is tangent at C_2 and D_2, and also E_1, F_1 and E_2, F_2 are collinear with L.

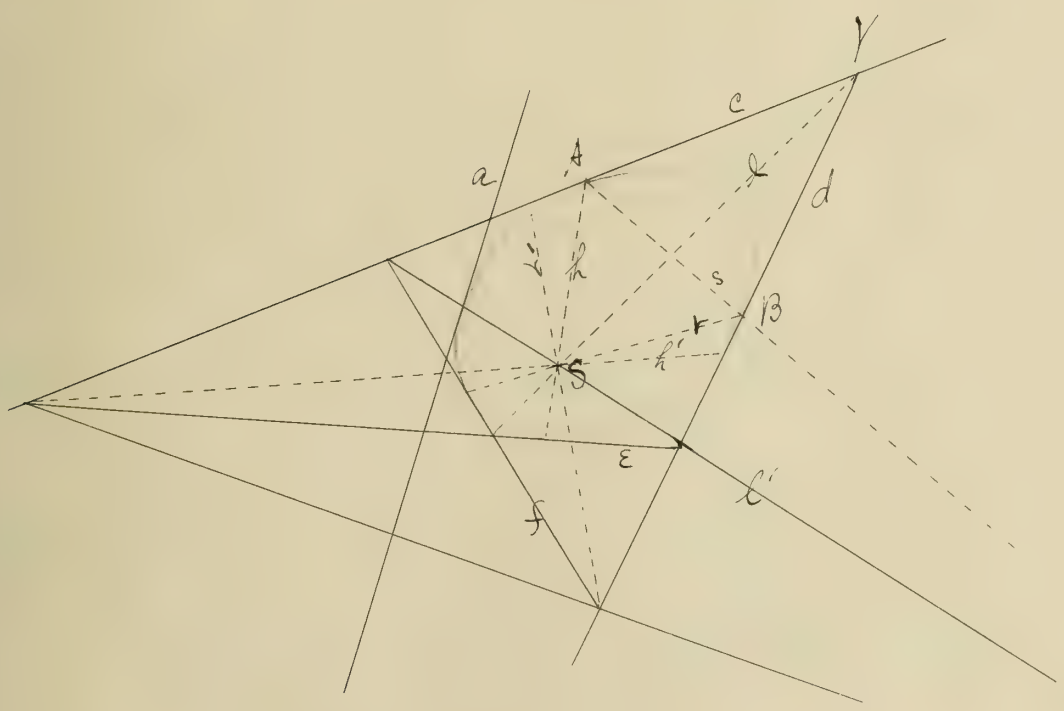
Now X and X' are a pair of correspondents on s. that coincide with a pair of points determined on s by the two projective pencils $A\,C_1$ and $A\,E_1$. X and X' determine the points C, D, E, F by construction, which satisfy the same conditions as C_1, D_1, E_1, F_1. Therefore the conic determined by A, C, D, E, F is tangent to a and b at C and D, and passes through the two imaginary points on s.

In the construction the line could have been drawn through L' instead of through L, since L and L' can be interchanged. Therefore two pairs of points X and X' can be determined which ~~will~~ be common to the involution and the two projective pencils through A. Therefore there are four solutions to the problem.

Problem 8a.

To construct a conic passing through two real points and tangent to one real and two imaginary lines.

Given. The real points A and B and the real line a and the imaginary lines α and α' which are the fixed rays of an elliptic involution at S.



Construction. Join A and B by means of the line s. Join S with A and B by means of the lines h and r. Find the correspondents h' and r' of h and r, with respect to the given involution at S. There is one pair of rays of the involution which divide h and r harmonically; let l and l' be that pair of rays. Choose any point P_1 on l and join it with A and B, by the lines c_1 and d_1. Find the intersection of r' and d_1, and of h' and c_1, join these points by means of the line p, which passes through the intersection of s and l'. Find the intersection of l' with d_1, and of c_1 with h', join these points by means of the line z_1. Find the intersection of l' with c_1 and of h' with d_1, join these points by means of the line f_1. The point of intersection of z_1 and d_1 lies on l. Find the intersection of a and c_1 and z_1, join these points with S by means of the lines ~~X'_1 and X~~ x'_1 and x. Choose on l a second point, join with A and B by means of the lines c_2 and d_2 and repeat construction. The intersections of a with c_2 and z_2 ~~with~~ determines a second pair of rays x'_2 and x.

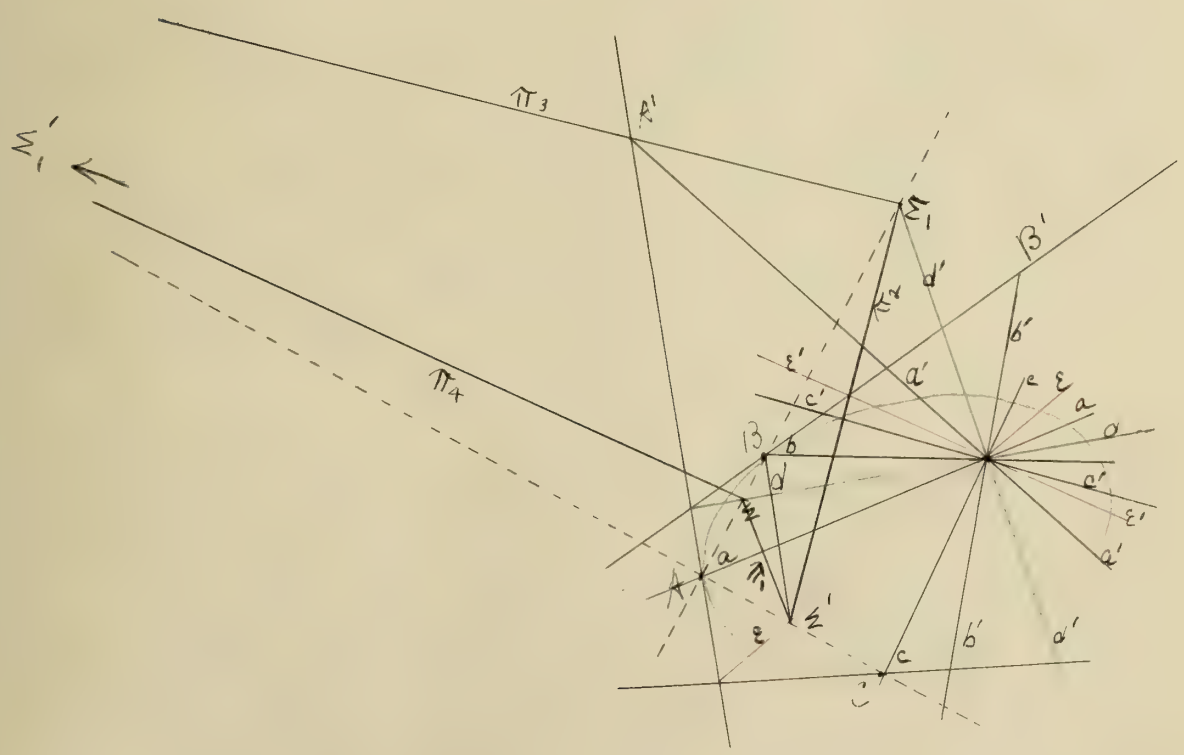
The rays x_i and x_i' determine two projective pencil ranges at S_3 as P moves along l. Find that pair of rays of the range which coincides with a pair of rays of the given involution. Call them x and x'. Find the intersection of x with a, join this point to A by means of the line c, and the intersection of x' with a, join this point with B by means of the line d. Find the intersection of l' with d and of h' with c, join these points by means of the line e. Find the intersections of l' with c and h' with d, and join these points by means of the line f. The five lines a, b, c, d, e, f are tangent to a conic satisfying the given conditions. There are four solutions for a reason similar to that in problem 8.

Proof. The proof is the exact dualistic of problem 8.

Problem 9.

To construct a conic passing through three real points, and tangent to a pair of imaginary lines.

Given. The real points A, B, C, and a pair S. imaginary lines ~~aa~~ α and α', as the fixed rays of an elliptic involution through S.



Construction. Join S with A, B, and C by the lines a, b, and c, respectively, and construct the rays a', b', c' conjugate to a, b, c with respect to the given involution at S. Construct through S that pair of rays which shall divide a b harmonically, also that pair which divides a c harmonically. Let AB intersect d at Σ and d' at Σ_1, ~~Connect~~ and let AC intersect Σ at Σ' and Σ' at Σ_1'. Connect $\Sigma\,\Sigma'\,\Sigma_1\,\Sigma_1'$ in the order named. Denote these four lines by $\pi_1\,\pi_2\,\pi_3\,\pi_4$ and extend them until they meet a', b', c', respectively. Connect $(\pi_1\,a')$, $(\pi_2\,a')$, $(\pi_3\,a')$, $(\pi_4\,a')$ with A and there are obtained four tangents at A to four conics satisfying the given conditions, all solutions real.

Proof. It is required to prove that the lines AA' is tangent at A to ~~the~~ a conic satisfying the given conditions. The tangent at A must pass through the pole of the ray a. The pole of the ray a is on the ray a'; it must also lie on the polar of S, with respect

to the conic. π_3 is the polar of S, since Σ_1 is the pole of d, and Σ_1' is the pole of z, therefore π_3, the line joining Σ_1 and Σ_1', is the polar of the intersection of d and z, which is S. Therefore the point A' which is the intersection of a' with π_3 is the pole of the ray a. Therefore the line AA' is tangent to a conic satisfying the given conditions. It can be proved in the same way that BB' and cc' are tangent to the a conic satisfying the given conditions.

Problem 9a.

To construct a conic passing through two imaginary points and tangent to three real lines.

Given. The imaginary points o and o', which are the fixed points of an elliptic involution on s, and the three real lines a, b, c.

Construction. Let s intersect a at A, b at B, and c at C. Find the points A', B', C' which

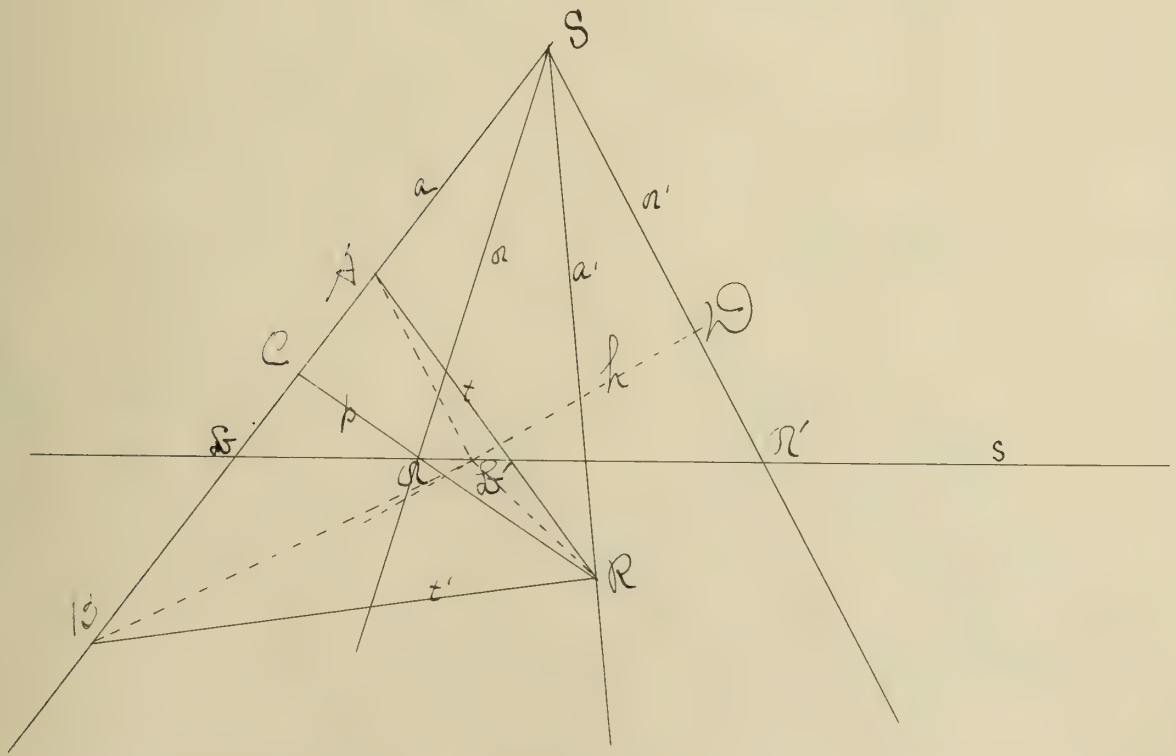
the correspondents of A, B, C on s with respect to the given involution. Find on S that pair of points, D and D', of the involution which divide A B harmonically; also find the points E and E' which divide A C harmonically. Join the intersection of a and b with D by the line ε, and with D' by the line ε_1. Join the intersection of a and c with E by the line ε', and with E' by the line ε_1'. Find the intersections of $\varepsilon\ \varepsilon'\ \varepsilon_1\ \varepsilon_1'$ in the order named. Denote these four points by $\pi_1, \pi_2, \pi_3, \pi_4$) join them with A', B', C', respectively. Find the intersections of $(\pi_1 A')\ (\pi_2 A')\ (\pi_3 A')\ (\pi_4 A')$ with a and we shall obtain four points on a, which are points of tangency of four conics satisfying the given conditions. The solutions are all real.

Proof. The proof is the exact dualistic of problem 9.

Problem 10.

To construct a conic passing through one real and two imaginary points and tangent to two imaginary lines.

Given the real point A and the imaginary points δ and δ' which are the fixed points of an involution on s, and the imaginary lines β and β' which are the fixed rays of an involution through S.

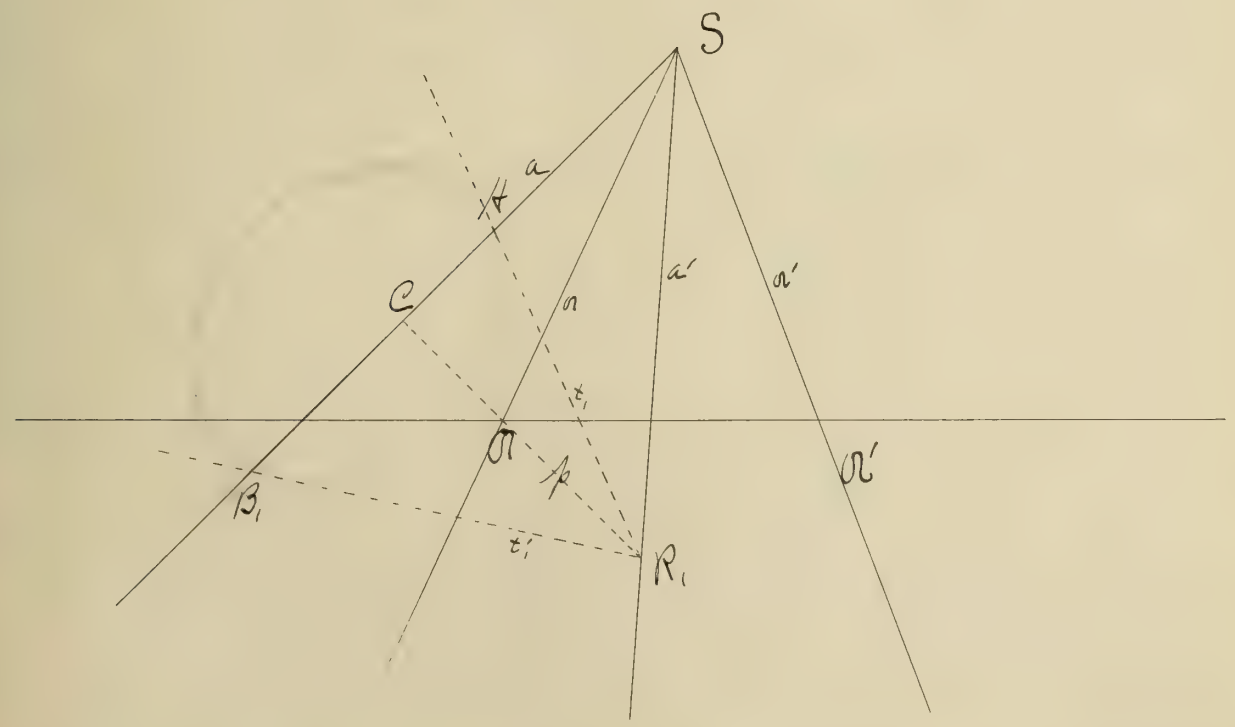


Construction. If we connect the points of the given involution on s with S we have two involutions of rays through S. Since these are both elliptic, they must have a common pair of real correspondents, a and a', cutting s in A and A' respectively. These correspondents we can construct.

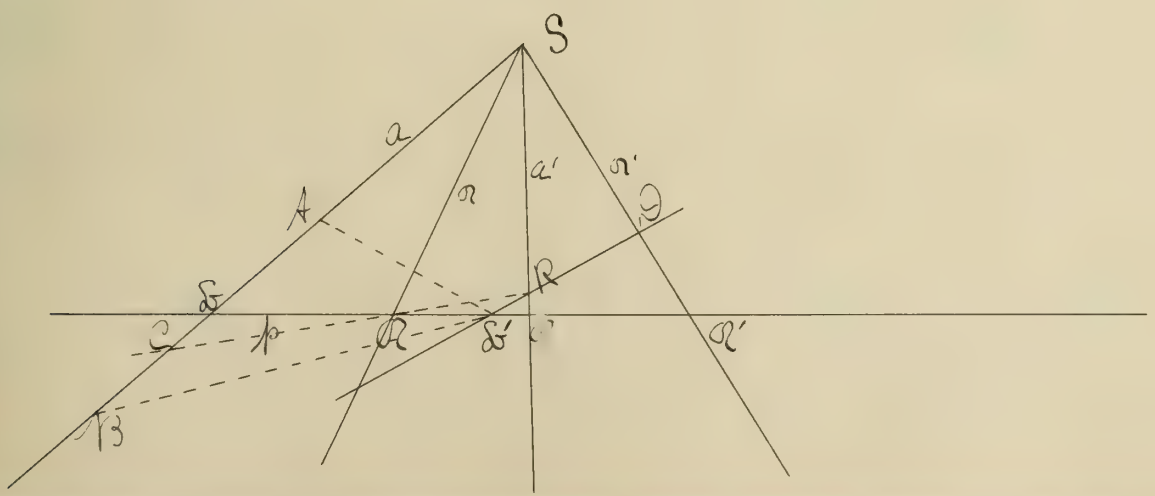
Now join S with A and let SA = a, meet s at L. Let a' be the ray through S conjugate to a, and let L' be the point on s conjugate to L. a' does not pass through L'. In a' choose any point R_1 and connect R_1 with R by means of p_1 and with A by means of t_1. Construct through R_1 a ray t_1' which together with t_1 divides a' and p_1 harmonically. Let this ray cut SA at B_1. Join L' with R_1, A, and B_1, and construct through L' that ray h_1 which with L L' divides L'A and L'B_1 harmonically. The rays L'R_1 and h_1 determine two points P_1 and P_1' on a'. ~~The~~ Choose another point R_2 on a', and go through same construction. The points P_1, P_2, P_3 ... and P_1', P_2', P_3' ... form two pro-

ject. Point rows on s s'. Find their double points Q and Q'. Connect Q with Q' cutting s' in R. Then RS is the "polar" of S with respect to the required conic. The conic must also pass through A and B and be tangent to RA and RB at A and B. It is thus determined. In a like manner the point Q' enables us to determine a second conic satisfying the conditions of the problem. In all there are four solutions to the problem.

Proof.



We must first prove that the conic through A and B tangent to RA and RB at A and B is also tangent to the double rays of the involution at S. Let RR = p cut AS at C. Then since p is the polar of S with respect to the conic, it cuts the conic in the points of contact of tangents from S. Furthermore R is the pole of a with respect to the conic. Hence the two tangents from S divide a and a' harmonically. Furthermore a' is the polar of R. Hence by a similar reasoning as above the tangents from S divide R and R' harmonically. These tangents are then for the double rays of the involution at S. We must now prove that the conic we have determined also passes through the two imaginary points on s.



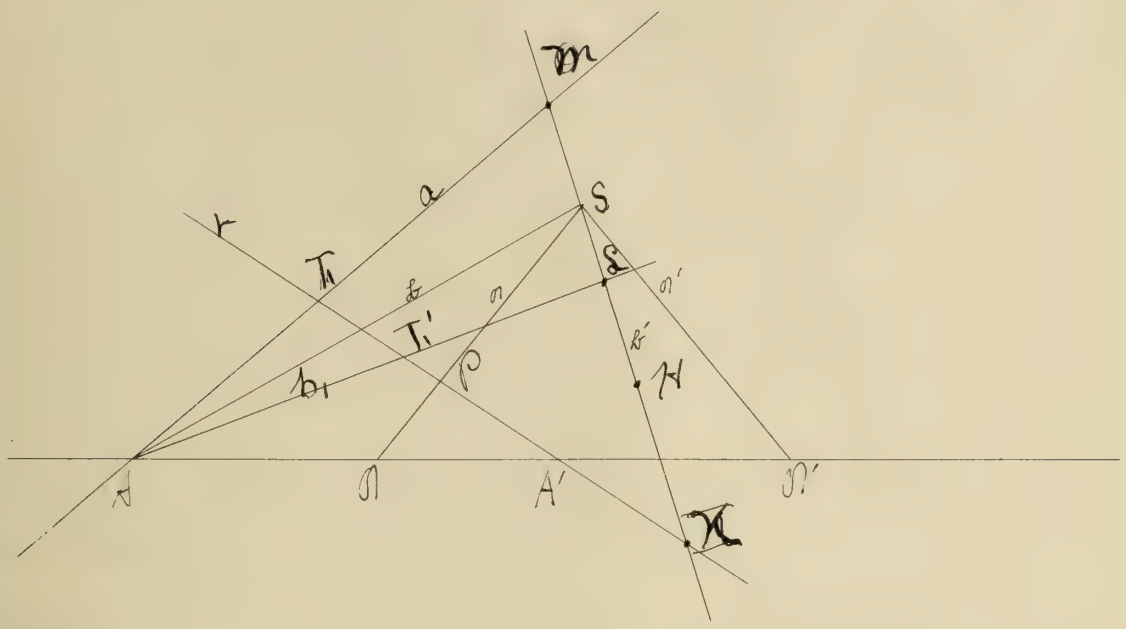
∴ It follows from the construction that the pole Ω with respect to the conic lies on h, and also on h', hence it is their intersection D. Now since the polar of D passes through R' and the polar of the correspondent R passes through D, it follows that the imaginary points in which the conic meets S must divide $R R'$ harmonically. Similarly since the polar of D passes through L and the polar of L' passes through D, it follows that the two points in which the conic meets S must divide $L L'$ harmonically. Hence these points are the double points of the given involution on S.

Problem 10a.

To construct a conic tangent to one real line and two imaginary lines, and passing through two imaginary points.

Given. The real line a, the imaginary lines α and α', which are the fixed lines of an elliptic involution at S, and the imaginary

points β and β' which are the fixed points of an elliptic involution on s.



Construction. If we let the ray ξ the given involution at S intersect s, we have two involution points on s. Since these are with the elliptic, they must have a common pair of points, R and R', which determine the rays r and r' at S, respectively. Now find the intersection, A_1, of s and a, and join A_1 with S by b. Let A' be the point on s corresponding to A, and let b' be the ray through S conjugate to b. A' does not lie on b'. Through A' draw any line r_1 and let r_1 meet R at P, and meet a at t_1. Construct ~~through~~ on r_1 a point t_1' which together with t_1 divides $A'P_1$ harmonically. Let

T_1' and A be joined by p_1. Find the intersection of p_1 with r_1, a, and b_1, and construct on b that point t_1, which with S divides the intersection of p_1 and a, m and b and b_1, Σ harmonically. The points X and A_1 determine two rays, p_1 and p_1', through S'. The rays $p_1, p_2, p_3 \cdots$ and $p_1', p_2', p_3' \cdots$ form two projective pencil ranges through S'. Find their double rays g and g'. ~~Connect~~ Find the intersection of g with b', joining this point with A' by means of r. Then P is the pole of S with respect to the required conic. The conic must also be tangent to a and b, and pass through T and T' on a and b. It is thus determined.

In a like manner the ray g' enables us to determine a second conic satisfying the conditions of the problem. In all there are four solutions of this problem.

Proof. The proof is the exact dualistic of problem 10.

Chapter 3.

In this chapter two constructions as given by F. Ruth and F. Path are given. They were with students of Stolz, so the methods are practically his. Although the constructions are based on the same principle as were those of Rossfeld no mention is made of his work. The analytical solution, as well as the synthetic construction, ~~are~~ one case shown as presented by F. Path. In the following German script will be used to denote imaginary elements, and English to denote real elements.

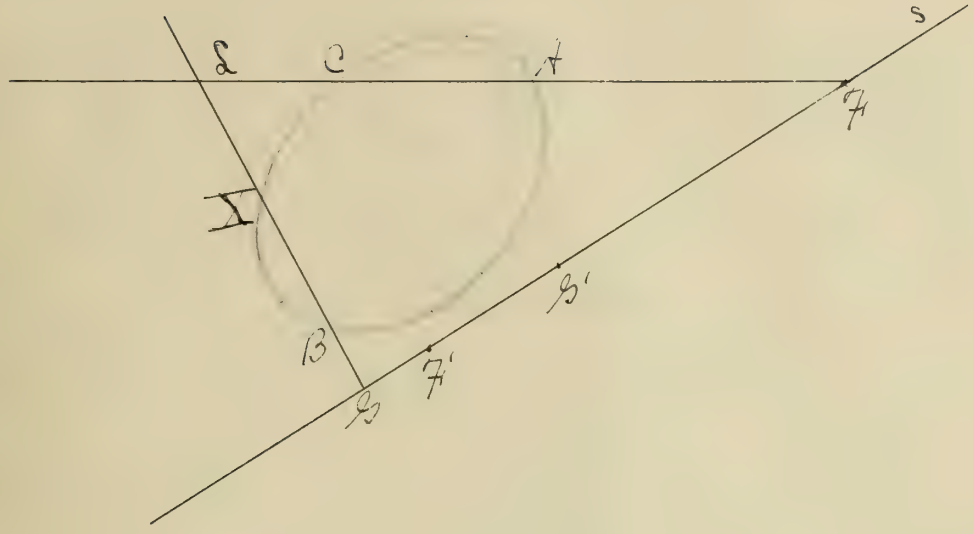
§ I

Problem I.

To construct a conic when three real points and two imaginary points are given.

Given. The real points A, B, C, and the ~~real~~ imaginary points α and α', which are the fixed points of an elliptic involution on s.

Construction. Join A with C and let AC meets in F. Draw any line through B and let it meets in G. Find the correspondents of F, and G', of F and G, with respect to the given involution on s. By Pascal's Theorem construct a conic through the five real points A, B, C, G, F'. It will intersect the line BG in X. This is a point of the desired conic. Any other point may be found by choosing other lines through B.



Prop. Let AC and BS meet at Σ. Then there is an elliptic involution in which $\alpha\alpha$, $\alpha'\alpha'$; SS'; HH' are correspondents. From this it follows that $\alpha\alpha'$, SH, $S'H'$ are also correspondents & an involution on S; since

$$\frac{\alpha H' \times \alpha H}{\alpha H \times H'\alpha'} = 1 = \frac{\alpha' H' \times \alpha H}{\alpha' H \times H'\alpha}$$

$$\frac{\alpha S' \times \alpha S}{\alpha S \times S'\alpha'} = -1 = \frac{\alpha' S' \times \alpha S}{\alpha' S \times S'\alpha}$$

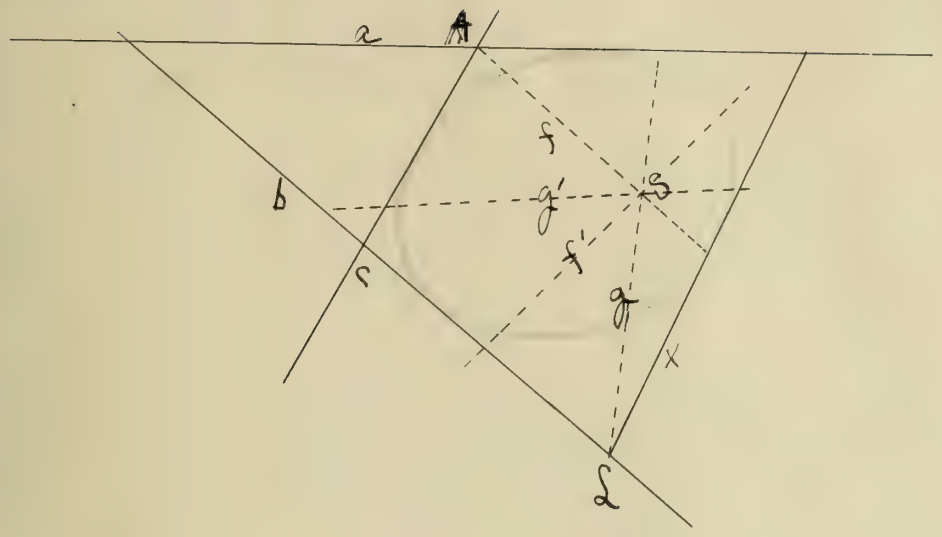
The four points A, C, S', H' determine a system of conics which determine an involution on BS. Consider the lines AC and S as one of

the conics; hence G and Σ are correspondents of the involution on BG. The conic through A, B, C, G', F' cuts the line BG in B and X, hence B and X are correspondents. The system of conics through A, B, C, X also determines an involution on s. Consider the lines AC and BG as one of the conics; it cuts the line s at G and F, therefore G and F are correspondents. The conic through A, B, C, G', X cuts the line in G and F', therefore G' and F' are correspondents. Hence a conic through A, B, C, X, α cuts the line s in α and α'; since α and α' are correspondents. Therefore X is a point of the desired conic.

Problem Ia.

To construct a conic tangent to three real lines and to two imaginary lines.

Given. The three real lines a, b, c, and the imaginary lines α and α', which are the fixed rays of an involution at S.



Construction. Let a and c meet at A. Join A with S by means of the line f. Choose any point L on b and join L with S by means of the line g. Find the rays f' and g' which are correspondents to f and g with respect to the given involution at S. By means of Brianchon's Theorem construct a conic which is tangent to the real lines a, b, c, f', g'. This conic is tangent to the line x through L, the line x is also tangent to a conic satisfying the given conditions. By choosing the point L in different positions no number of tangents may be obtained.

Proof. At S there is an involution in which $\alpha\alpha$, $\alpha\alpha'$; ff'; gg' are correspondents. From this it follows that $\alpha\alpha'$, gf, $g'f'$ are also correspondents in an involution at S, since

$$\frac{\alpha f' \times \alpha f}{\alpha f \times f'\alpha'} = -1 = \frac{\alpha' f' \times \alpha f}{\alpha' f \times f'\alpha}$$

$$\frac{\alpha g' \times \alpha g}{\alpha g \times g'\alpha'} = -1 = \frac{\alpha' g' \times \alpha g}{\alpha' g \times g'\alpha}$$

The four tangents a, c, g', f' determine a system of conics which determine an involution of rays at L. The points A and S determine one of these conics, it is tangent to the rays l and g, hence l and g are correspondents. The conic tangent to a, b, c, g', f' is tangent to the lines b and x through L, hence b and x are correspondents. The system of conics tangent to a, b, c, x determines an involution at S. Consider the points A and L as one of the conics, it determines the rays g and f, hence g and f are correspondents. The conic tangent to a, b, c, g', x

determines the rays g' and f', hence g' and f' are correspondents. Therefore a conic tangent to a, b, c, X, α determines the rays α and α', since α and α' are correspondents, therefore the line X is a tangent to the desired conic.

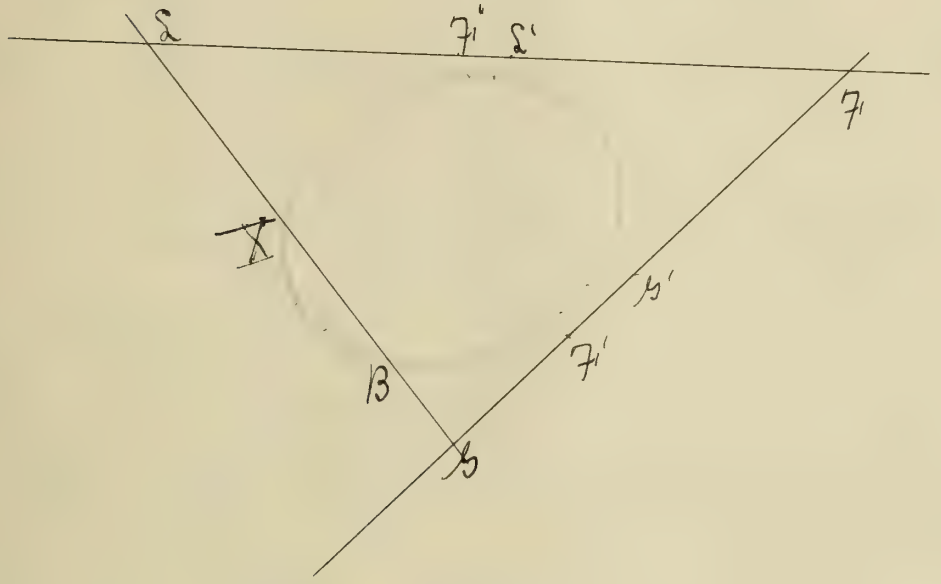
Problem 2.

To construct a conic passing through one real and two pair of imaginary points.

Given. The real point B, and the imaginary points α and α' and β and β' which are the fixed points of elliptic involutions on s and s_1, respectively.

Construction. Let any line through B meet s in G and s_1 in Σ. Find the correspondents, G' and Σ' of G and Σ, on s and s_1, with respect to the given involutions. Find the correspondents T_1' and T_1'' of T_1, the intersection of s and s_1, with respect to the given involutions. Through the five real points $B, G, G', T_1, \Sigma, \Sigma', T_1', T_1''$, construct a conic, it will cut the line $B G$ in

B and X. Now X is a point of the desired conic. By choosing the line through B in different positions any number of points may be found.



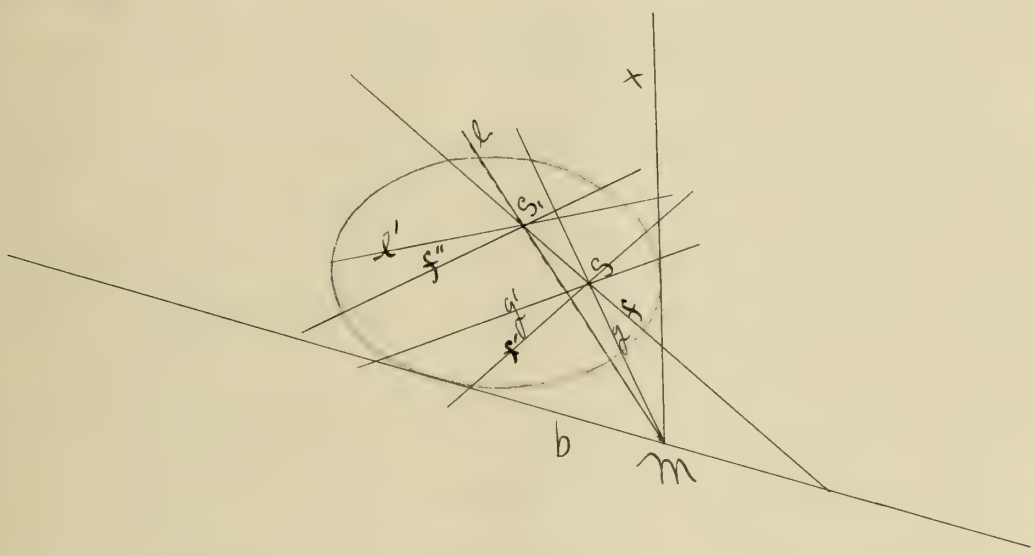
Proof. On s there is an involution in which $\alpha\alpha$, $\alpha'\alpha'$, $\beta\beta$, $\gamma\gamma$, $\gamma'\gamma'$ are correspondents, and on s_1 an involution in which $\beta'\beta'$, $\beta\beta$, $\alpha\alpha'$, $\gamma\gamma''$ are correspondents. From this it follows, as in problem I, that there exists involutions in which $\alpha\alpha'$, $\beta\gamma$, $\beta'\gamma'$ are correspondents, and $\beta\beta'$, $\alpha\gamma$, $\alpha'\gamma''$ are correspondents. The system of conics through $\alpha'\gamma''$, $\beta'\gamma'$ ~~with~~ determines an involution on

$B_1 G$. Consider s and s_1 as one of the conics, it cuts $B_1 G$ in G and L; therefore G and L are correspondents. The conic through L', F'', G', F', B cuts the line $B_1 G$ in B and X; therefore B and X are correspondents. The system of conics through L', F'', X, B, determine an involution on s. Consider the lines s_1 and $B_1 G$ as one of the conics; it cuts s in G and F; therefore G and F are correspondents. The conic through X, B, F'', L', F' cuts the line s in F' and G; therefore F' and G' are correspondents. Since the conic through X, B, F'', L', α cuts the line s in α and α', because α and α' are correspondents. It may be proved in the same way that the conic through X, B, F', G', β will cut the line s_1 in β and β', since β and β' are correspondents. Therefore the point X is a point of the desired conic. By choosing the line through B in different positions any number of points may be found.

Problem 2a.

To construct a conic tangent to a real line and to two pair of imaginary lines.

Given: The real line b, and the imaginary lines α and α'; and β and β', which are the fixed ~~foci~~ rays of involutions through S and S_1.



Construction: Join S and S_1 by means of the line f. Let f'' be the correspondent of f in the involution at S and f' the correspondent of f in the involution at S_1. Choose any point, M, on b and join M with S by means of the line l and to S_1 by means of the line g. Find the correspondents l' and g' of l and g. By

means of Brianchon's Theorem construct a conic tangent to the five real lines b, f'', f', g', l'. This conic ~~will be~~ is tangent to the line x through the point M. Any number of tangents can be drawn by choosing the point M in different positions.

Proof. At S there is an involution in which $\alpha\alpha''\alpha^*, \alpha\alpha_1, gg'; ff'$ are correspondents and at S_1 there is an involution in which $\beta\beta; \beta'\beta', ll', ff''$ are correspondents. From this it follows, as in problem I a, that there exist involutions at S and S_1, in which $\alpha\alpha'; gf, g'f'$ ~~and~~ are correspondents, and $\beta\beta', lf, l'f''$ are correspondents. The system of conics tangent to l', f'', g', f' determines an involution at M. Consider the points S and S_1 as one of the conics, it determines the rays g and l at M, therefore g and l are correspondents. The conic tangent to l', f'', g', f', b determines the rays b and x at M, therefore b and x are correspondents. The

system ∞^5 conics tangent to l', f'', x, b determines
in involution at S. Consider the point S_1
and M as one of the conics; it determines
the ray g and f at S, hence g and f are
correspondents. The conic tangent to x, b, f'', f', l'
determines the rays f' and g'; hence f' and g' are
correspondents. Therefore the conic tangent to
x, b, f'', l', α determines the rays α and α' at S,
since α and α' are correspondents. Similarly
the conic tangent to, x, b, f', g', β determines the
rays β and β' at S_1, since β and β' are corres-
pondents. Therefore x is a tangent to the
required conic.

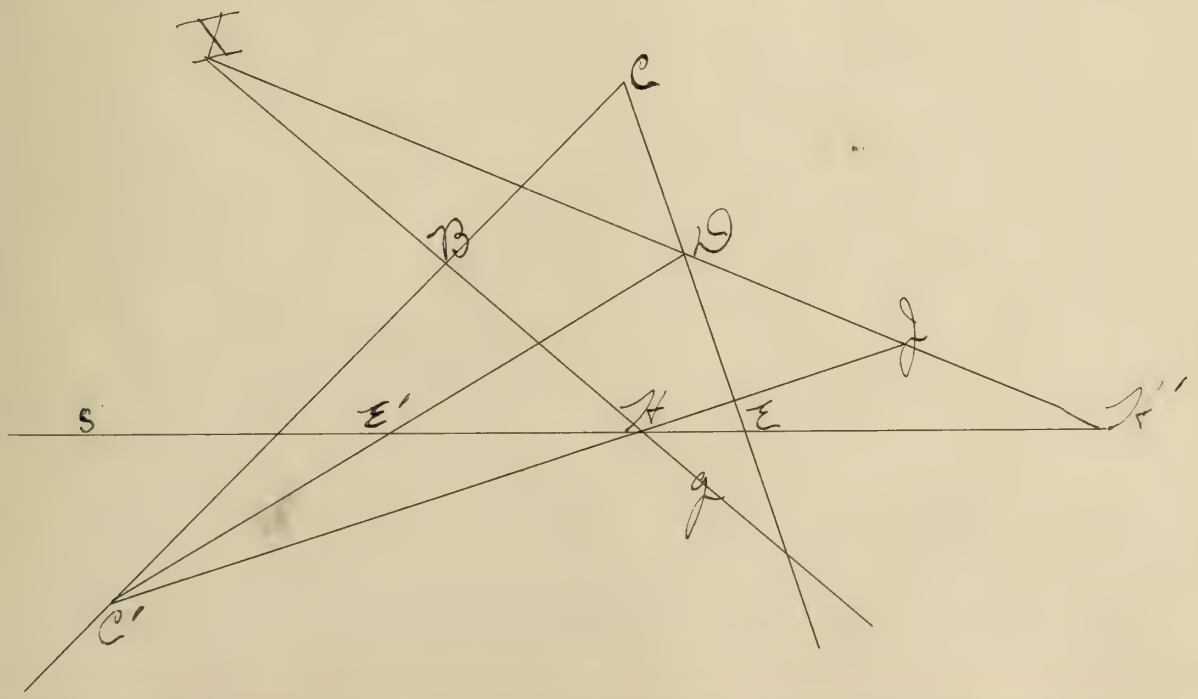
§ 2.

In the following is given the construction, and also the analytical solution, of one case, as given by F. Spath. The constructions used by F. Ruth and F. Spath are practically the same.

Problem I.

To construct a conic passing through three real points and two imaginary points.

Given. The three real points B, C, D, and the imaginary points S and S', in s.



Construction. Any line g through B cuts s in F, $C D$ cuts s in E. Find the correspondents F' and E' of F and E with respect to the given involution on s. Let DE' and BC intersect at C', and FC' and $F'C$ at J. The point where the line DJ cuts g is a point of the desired conic.

The analytical expressions of the given elements are as follows:

O	$X_1 = 0$	$X_3 + \mu i X_2 = 0$
O'	$X_1 = 0$	$X_3 - \mu i X_2 = 0$
B	$X_2 = 0$	$X_3 = 0$
C	$X_2 = 0$	$X_3 + \alpha X_1 = 0$
D	$\rho X_1 = \eta_1$;	$\rho X_2 = \eta_2$; $\rho X_3 = \eta_3$; $\eta_1 \geq 0$, $\eta_2 \leq 0$
g	$X_3 + \beta X_2 = 0$	

We wish to find the analytical expressions for the following points:-

$\left.\begin{matrix} \mathfrak{A}\mathfrak{A}' \\ \mathfrak{B}\mathfrak{X} \end{matrix}\right\} H \qquad \left.\begin{matrix} \mathfrak{B}\mathfrak{C} \\ \mathfrak{L}\mathfrak{D}\mathfrak{A} \end{matrix}\right\} \mathfrak{K} \qquad \left.\begin{matrix} \mathfrak{C}\mathfrak{A} \\ \mathfrak{L}\mathfrak{D}\mathfrak{X} \end{matrix}\right\} \mathfrak{J}$

$\left.\begin{matrix} x_1 = 0 \\ x_3 + \beta x_2 = 0 \end{matrix}\right\} \mathfrak{H} \qquad \left.\begin{matrix} x_2 = 0 \\ (\eta_3 + \mu_i \eta_2) x_1 - \eta_2 (x_3 + \mu_i x_2) = 0 \end{matrix}\right\} \mathfrak{K}$

$\mathfrak{H}\mathfrak{K}$ $(\eta_3 + \mu_i \eta_2) x_1 - \eta_1 (x_3 + \beta x_2)$

$\equiv [(\eta_3 + \mu_i \eta_2) x_1 - \eta_1 (x_3 + \mu_i x_2)] + \eta_1 (\mu_i - \beta) x_2 = 0 \Big\} \mathfrak{J}$

$\mathfrak{C}\mathfrak{A}'$ $\alpha x_1 + x_3 - \mu_i x_2 = 0$

$\mathfrak{L}\mathfrak{D}\mathfrak{J}$ is a real line, since the coefficient of μ_i in the equation is

$$\eta_1 \eta_2 [(\alpha x_1 + x_3) + \beta x_2] - \alpha \eta_1 \eta_2 x_1 - \eta_1 \eta_2 (x_3 + \beta x_2) = 0$$

$\mathfrak{L}\mathfrak{D}\mathfrak{J}\mathfrak{X}$ passes through the intersection of the straight lines whose equations are given by the real parts of the following equations

$$(\mu_i - \beta)(\alpha x_1 + x_3 - \mu_i x_2) = 0$$

$$(\alpha \eta_1 + \eta_3 - \mu_i \eta_2)[(\eta_3 + \mu_i \eta_2) x_1 - \eta_1 (x_3 + \beta x_2)] = 0$$

The real and imaginary parts of the first equation determine, in involution at P, two conjugate rays. The latter of these cuts

$$\alpha x_1 + x_3 + \beta x_2 = 0$$

cuts the straight line $x_1 = 0$ in A'. The real part of the first equation also determines the straight line C A'.

The real and imaginary part of this equation determines the conjugate ray, h and h', of the involution at A' K. This involution we may obtain from DA, by joining the intersection of DA and BC with A'. Or instead we join A' with the intersection of AB and the line whose equation is given by the real or imaginary part of

$$(\alpha \eta_1 + \eta_3 - \mu i \eta_2)[(\eta_3 + \mu i \eta_2) x_1 - \eta_1 (x_3 + \mu i x_2)] = 0$$

also gives h and h'. The imaginary part of this last equation may be written

$$\eta_2(\eta_1 x_3 - \eta_3 x_1) - (\alpha\eta_1 + \eta_3)(\eta_3 x_1 - \eta_1 x_2) = 0$$

$$(\alpha\eta_1 + \eta_3)x_2 - \eta_2(\alpha x_1 + x_3) = 0$$

This determines the straight line CHΣ, and the real part of that equation gives the line DΣ'. C' is the intersection of BC with DΣ'; and AC' is the line h, which locates the line DX in the above construction.

Bibliographie

1. August

Untersuchungen über das Imaginäre in der Geometrie. [Progr. d. Friedrichsschule Berl. 1872]

2. Boguslavsky, A. C.

Über M. Marie's Method der Darstellung der imaginären Elementen in der Geometrie [Arbeiten VIII Versamml. russ. Naturf. u. Aerzte. Abt. I, 16-26. (Russisch)].

3. ?. Hossfeld.

Construction des Kegelschnitts aus fünf zum Teil imaginären Curvenelementen. Dissertation, Jena (Neuenhahn, A.) 1882.

4. C. Hossfeld.

Über eine Aufgabe aus der projectiven Geometrie des Raumes. Construction der

Raum-curve dritter Ordnung aus imaginären Punkten. [Schlömich Z XXVIII 111-116]

5. C. Hossfeld.

Construction der Fläche zweiter Ordnung aus neun Punkten von denen acht imaginären sind. [Schlömich Z XXXIII 180]

6. Kötter, E.

Die Entwickelung der synthetischen Geometrie [Deutsche Math. Ver. Vol. 5 1-128] As to that & imaginaries in continuation of this article.

7. Lüroth

Das imaginäre in der Geometrie und das Rechnen mit Würfen. Darstellung und Erweiterung der v. Standt. schen Theorie. [Clebsch Ann VIII 145-214]

8. Montesanelli, O

Construzioni proiettive delle curve di second'ordine con elementi immaginari [Battaglini VIII. n. 131 S 1887].

9. Rainorino, A.

Gli elementi ~~d~~ immaginarii nella geometria. Monografia storica. [Batt G. 35, 242-254; 36, 317-345].

10. F. Rudt

Beitrag zur Construction der Kegelschnitte aus imaginären Elementen [Monatsh. f. Math. III 81-86].

11. K. Schoter.

Construction von Kegelschnittenlinien aus imaginären elementen auf Grund neuer Sätze der Polarentheorie Inaug. Diss. 1892 [05]

12. C. A. Scott.

The status of imaginaries in pure geometry [Am. M. Soc. Bul. (2), 6, 163-164]

13. F. Spalt.

Lineale Construction von Kegelschnitten aus theilweise imaginären Elementen [Monatschf. für Math. I, 1890, p 237-246]

14. Vergl. F. Elemente der projectiven Geometrie. Heft 2. p. 57. (Construction of conics with imaginary elements)

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